Lab1 : Convolution

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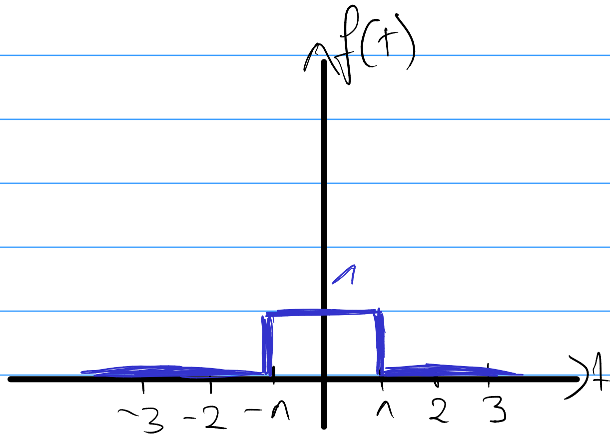
We want to convolve a rectangle with itself

a rectangle is defined as :

$$f(t) = \begin{cases} 1 & ; -1 \leq t \leq 1 \\ 0 & ; \text{otherwise} \end{cases}$$



1) the convolution is defined mathematically as:

$$g(x) = f(x) * h(x) = \int_{-\infty}^{+\infty} f(t) \cdot h(x-t)\, dt$$

In this case we have $f(t) = h(t)$

2) Let's define the support of each function

$f(t)$ is non zero in $-1 \leq t \leq 1$ $\qquad [-1, 1]$

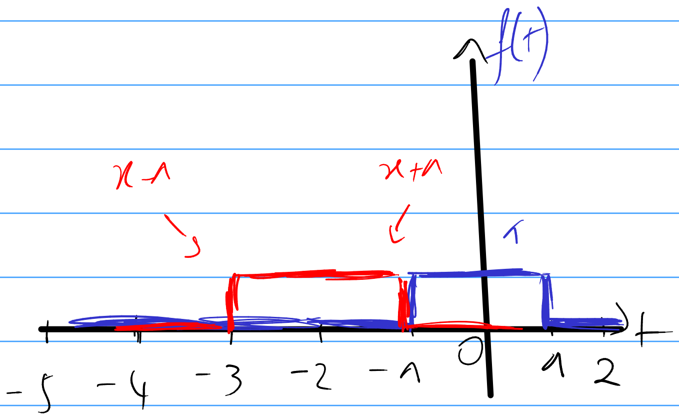
$f(x-t)$ is non zero in

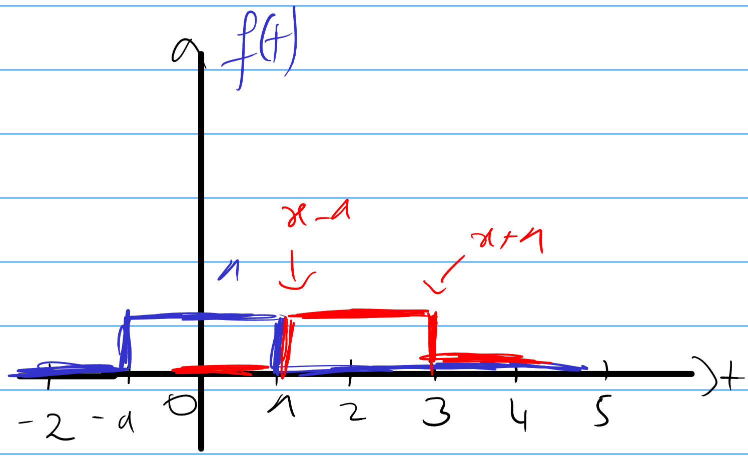
$$-1 \leq x-t \leq 1$$
$$-x-1 \leq -t \leq -x+1$$
$$x-1 \leq t \leq x+1$$

the support of $h(x-t)$ is $[x-1, x+1]$

3) Compute the convolution for each of the following cases :

a) No overlap





We have no overlap when : $x-1=-3 \Rightarrow x=-2$

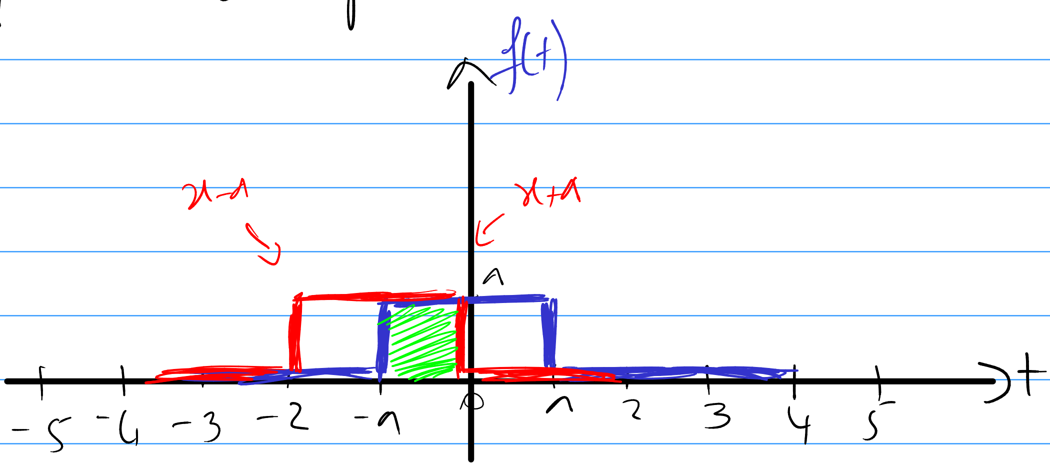
or $x+1=-1 \Rightarrow x=-2$

$\boxed{g(x)=0 \ ; \quad x\leq -2 \quad \text{or} \quad x\geq 2}$

and $x-1=1 \Rightarrow x=2$

or $x+1=3 \Rightarrow x=+2$

b) 1st partial overlap

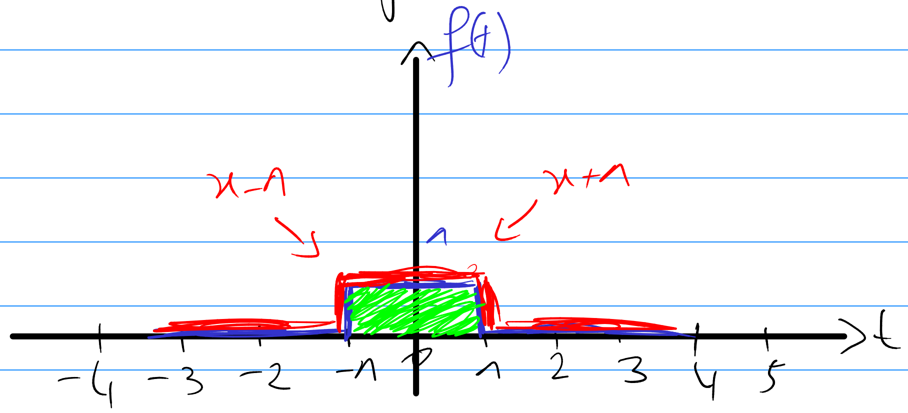


$$g(x)=\int_{-1}^{x+1} f(t).\,h(x-t)\,dt$$

$$=\int_{-1}^{x+1} 1\times 1\,dt=[t]_{-1}^{x+1}=x+1-(-1)=x+2$$

$$\boxed{g(x) = x+2 \quad \text{when} \quad -2 \leq x \leq 0}$$

Case 3) : Full overlap



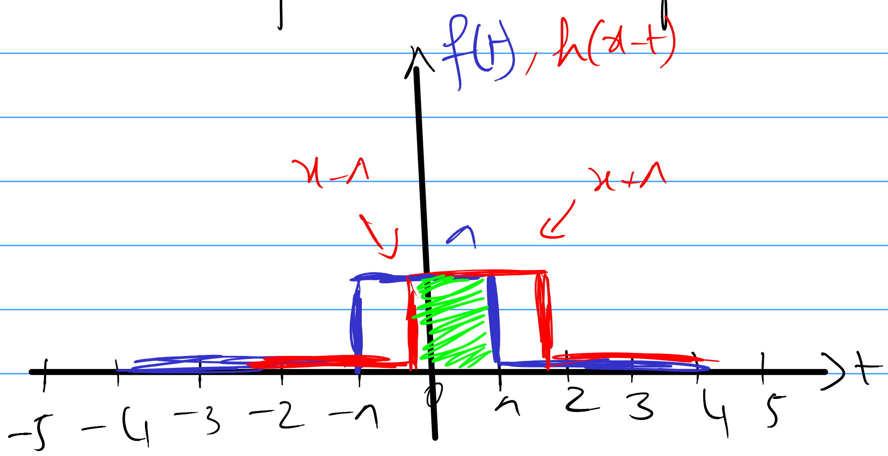
We have a full overlap at $x = 0$

$$g(x) = \int_{-1}^{1} f(t) \cdot h(x-t)\, dt$$

$$= \int_{-1}^{1} 1 \times 1 \, dt = [t]_{-1}^{1} = 1-(-1) = 2$$

$$\boxed{\text{at} \quad x=0 \quad g(x) = 2}$$

d) case 4 : 2nd partial overlap.



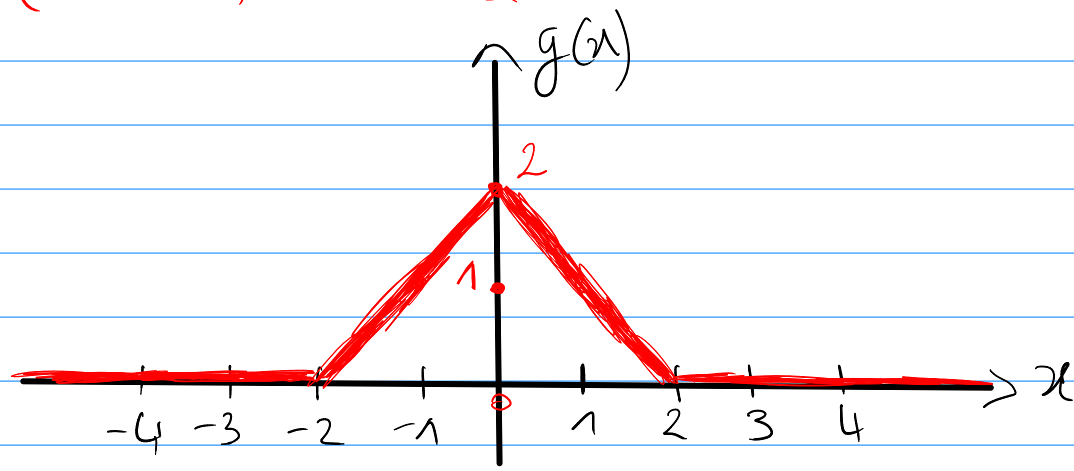
$$g(x) = \int_{x-1}^{1} f(t) \cdot h(x-t)\, dt$$

$$= \int_{x-1}^{1} 1 \times 1 \, dt = [t]_{x-1}^{1} = 1 - x + 1$$

$$= -x + 2$$

$g(x) = +2 - x$ when $0 \leqslant x \leqslant 2$

$$g(x) = \begin{cases} x+2 & ; \; -2 \leqslant x \leqslant 0 \\ 2 & ; \; x = 0 \\ -x+2 & ; \; 0 \leqslant x \leqslant 2 \\ 0 & ; \; \text{otherwise} \end{cases}$$



Lab1 : Convolution

we want to convolve a rectangular Pulse with a triangle, the convolution is defined as:

$$g(x) = f(x) * h(x) = \int_{-\infty}^{+\infty} f(t) \cdot h(x-t)\, dt$$

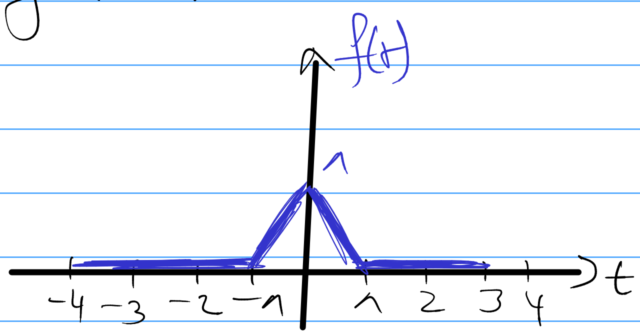
because the convolution is commutative meaning that:

$$f * g = g * f$$

$$g(x) = h(x) * f(x) = \int_{-\infty}^{+\infty} h(t) \cdot f(x-t)\, dt$$

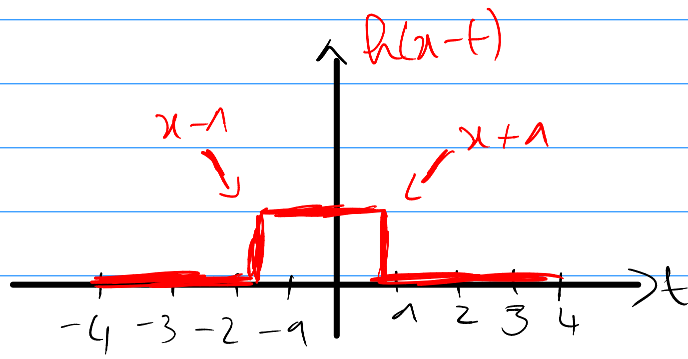
1) let's define and draw the functions :

$$f(t) = \begin{cases} 1+t & ; \; -1 \leq t \leq 0 \\ 1-t & ; \; 0 \leq t \leq 1 \\ 0 & ; \; \text{otherwise} \end{cases}$$



$$h(x-t) = \begin{cases} 1 & ; \; -1 \leq x-t \leq 1 \\ 0 & ; \; \text{otherwise} \end{cases}$$

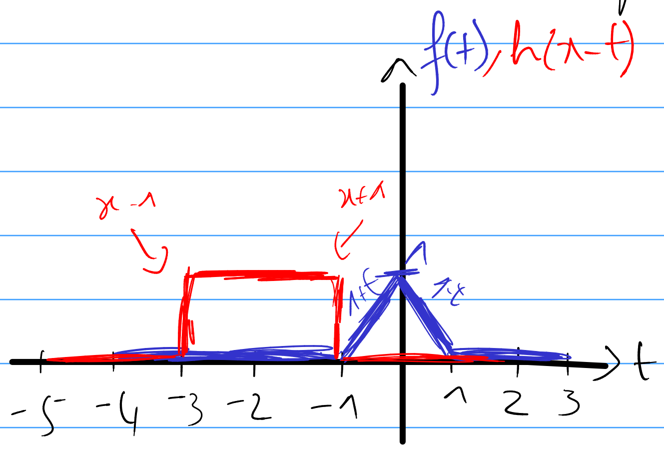
$$= \begin{cases} 1 & ; \; x-1 \leq t \leq x+1 \\ 0 & ; \; \text{otherwise} \end{cases}$$

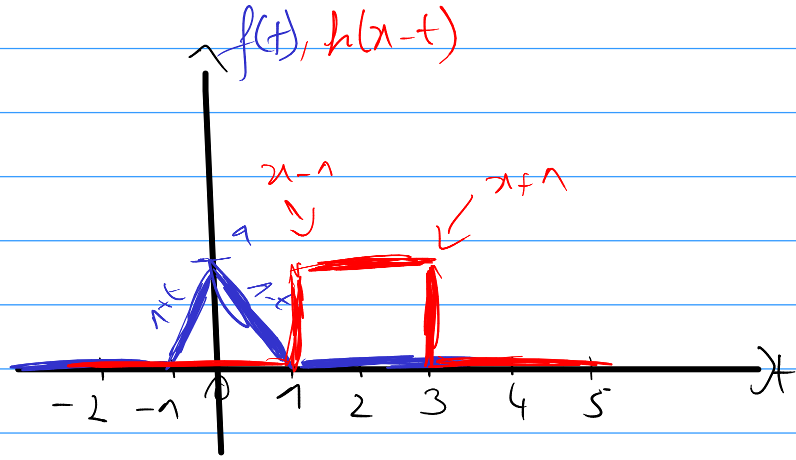


Triangle support : $[-1, 1]$
Rectangle support : $[x-1, x+1]$

2) Draw and compute the convolution for each of the following cases:

a) case 1) : No overlaps





$x-1 = -3$ and $x+1 = -1$
$x = -2$ and $x = -2$

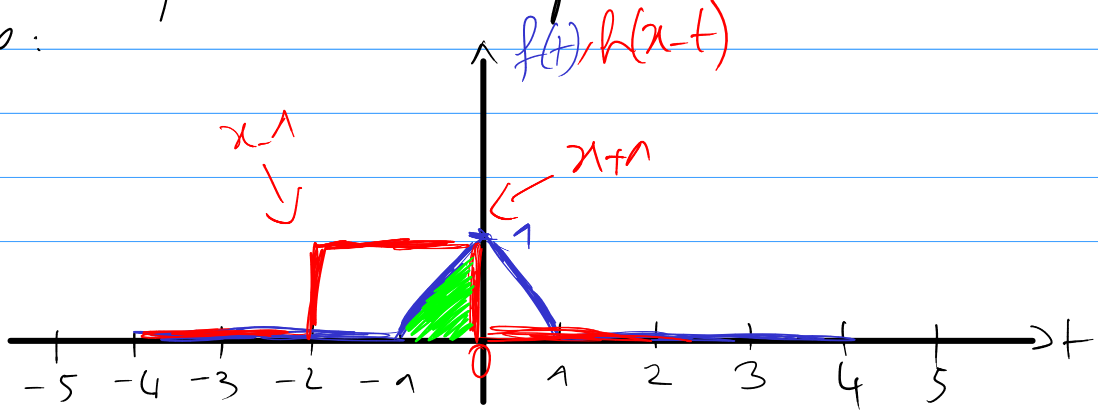
$x-1 = 1$ and $x+1 = 3$
$x = 2$ and $x = 2$

No overlaps when $x \leq -2$ and $x \geq 2$

$$\boxed{g(x) = 0}$$

b) Case 2: 1st partial overlap

Subcase 1b:



at $-2 \leq x \leq -1$

$$g(x) = \int_{-1}^{x+1} f(t) \cdot h(x-t)\, dt$$

$$= \int_{-1}^{x+1} (1+t) \cdot 1 \, dt$$

$$= \int_{-1}^{x+1} 1 \, dt + \int_{-1}^{x+1} t \, dt$$

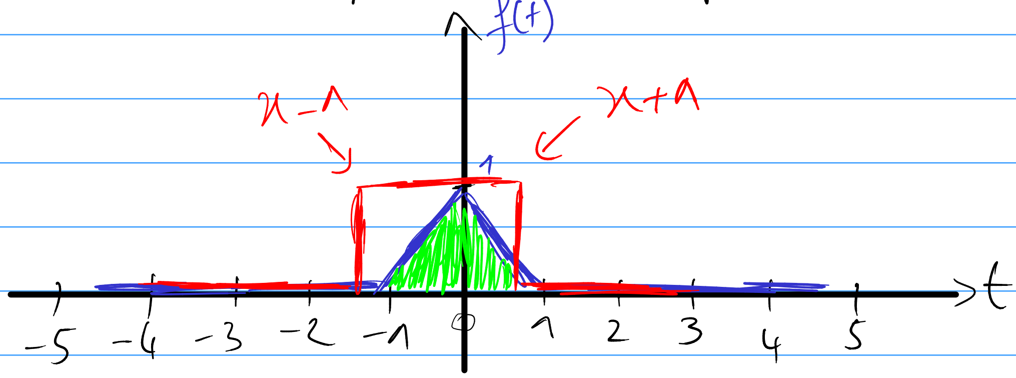
$$= \left[t \right]_{-1}^{x+1} + \left[\frac{t^2}{2} \right]_{-1}^{x+1}$$

$$= x+1 - (-1) + \frac{(x+1)^2}{2} - \frac{(-1)^2}{2}$$

$$= x+2 + \frac{1}{2}x^2 + \frac{2}{2}x + \frac{1}{2} - \frac{1}{2}$$

$$\boxed{g(x) = \frac{1}{2}x^2 + 2x + 2}$$

subcase 2b: 1st partial overlap



$$g(x) = \int_{-1}^{0} f(t) \cdot h(x-t)\, dt + \int_{0}^{x+1} f(t) \cdot h(x-t)\, dt$$

$$= \int_{-1}^{0} (1+t) \cdot 1 \, dt + \int_{0}^{x+1} (1-t) \cdot 1 \, dt$$

$$= \int_{-1}^{0} 1 \, dt + \int_{-1}^{0} t \, dt + \int_{0}^{x+1} 1 \, dt - \int_{0}^{x+1} t \, dt$$

$$g(x) = [t]_{-1}^{0} + \left[\frac{t^2}{2}\right]_{-1}^{0} + [t]_0^{x+1} - \left[\frac{t^2}{2}\right]_0^{x+1}$$

$$= 0 - (-1) + \frac{0^2}{2} - \frac{(-1)^2}{2} + x + 1 - 0 - \frac{(x+1)^2}{2} - \frac{0^2}{2}$$

$$= 1 - \frac{1}{2} + x + 1 - \frac{x^2}{2} - \frac{2x}{2} - \frac{1}{2}$$

$$g(x) = 1 - \frac{1}{2}x^2$$

when $-1 \leq x \leq 0$

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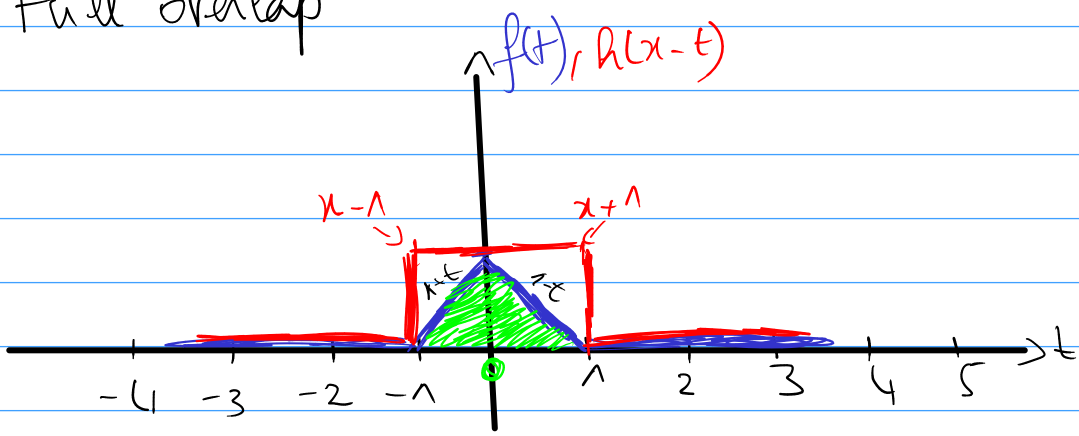
case 3 : Full overlap



we have a full overlap at $x=0$, so $g(x)$ is

$$g(x) = \int_{-1}^{0} f(t)\cdot h(x-t)\,dt + \int_0^1 f(t)\cdot h(x-t)\,dt$$

$$= \int_{-1}^{0} (1+t)\cdot 1\,dt + \int_0^1 (1-t)\cdot 1\,dt$$

$$= \int_{-1}^{0} 1\,dt + \int_{-1}^{0} t\,dt + \int_0^1 1\,dt - \int_0^1 t\,dt$$

$$= [t]_{-1}^{0} + \left[\frac{t^2}{2}\right]_{-1}^{0} + [t]_0^1 - \left[\frac{t^2}{2}\right]_0^1$$

$$g(x) = 0-(-1) + \frac{0^2}{2} - \frac{(-1)^2}{2} + 1 - 0 - \frac{1^2}{2} + \frac{0^2}{2}$$

$$= 1 - \frac{1}{2} + 1 - \frac{1}{2}$$

$$= 2 - 1 = 1$$

$$\boxed{\text{at } x=0 \ ; \ g(x) = 1}$$

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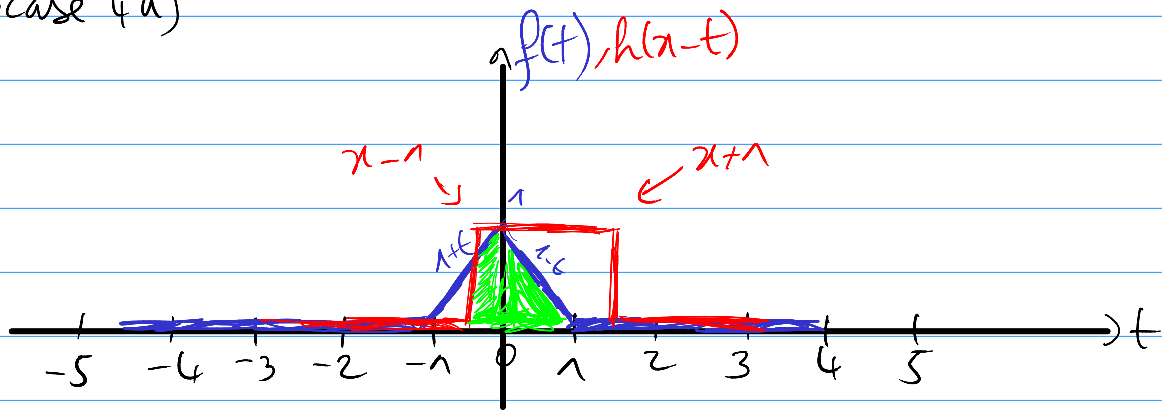
d). case 4 : 2nd partial overlap
subcase 4a)



$$g(x) = \int_{x-1}^{0} f(t)\cdot h(x-t)\,dt + \int_0^1 f(t)\cdot h(x-t)\,dt$$

$$= \int_{x-1}^{0} (1+t)\cdot 1\,dt + \int_0^1 (1-t)\cdot 1\,dt$$

$$= \int_{x-1}^{0} 1\,dt + \int_{x-1}^{0} t\,dt + \int_0^1 1\,dt - \int_0^1 t\,dt$$

$$= [t]_{x-1}^{0} + \left[\frac{t^2}{2}\right]_{x-1}^{0} + [t]_0^1 - \left[\frac{t^2}{2}\right]_0^1$$

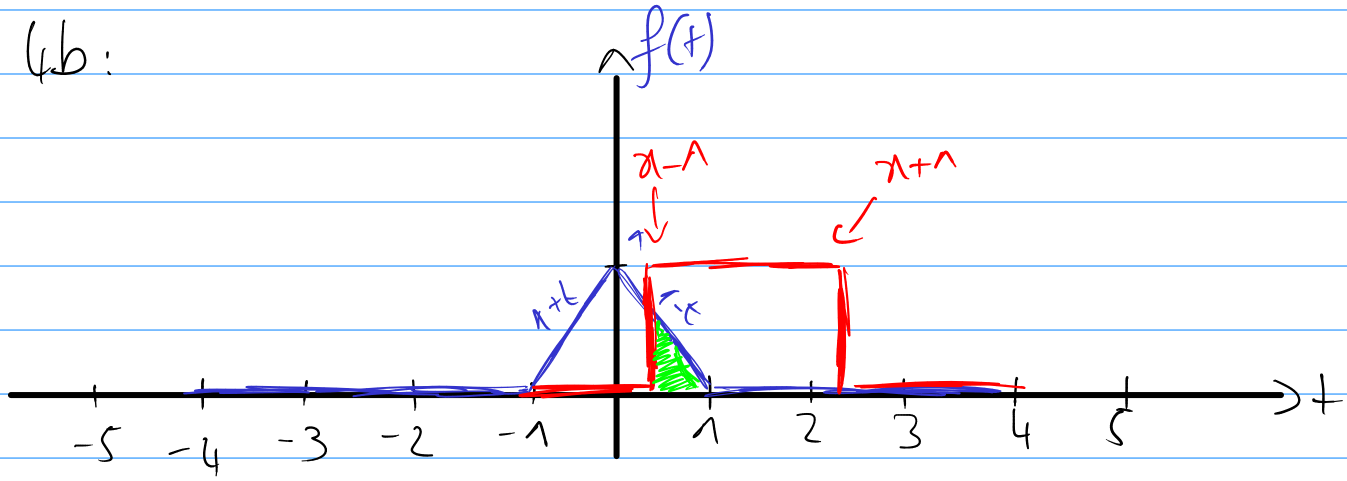
$$= 0-(x-1) + \frac{0^2}{2} - \frac{(x-1)^2}{2} + 1 - 0 - \frac{1^2}{2} + \frac{0^2}{2}$$

$$g(x) = -x+1 - \left(\frac{x^2-2x+1}{2}\right) + 1 - \frac{1}{2}$$

$$= -x+1 - \frac{1}{2}x^2 + \frac{2}{2}x - \frac{1}{2} + 1 - \frac{1}{2}$$

$$\boxed{g(x) = -\frac{1}{2}x^2 + 1 \quad \text{when} \quad 0 < x < 1}$$

Case 4b:



$$g(x) = \int_{x-1}^{1} f(t) \cdot h(x-t)\,dt$$

$$= \int_{x-1}^{1} (1-t) \cdot 1 \;dt$$

$$= \int_{x-1}^{1} 1\,dt - \int_{x-1}^{1} t\,dt$$

$$= \left[t\right]_{x-1}^{1} - \left[\frac{t^2}{2}\right]_{x-1}^{1}$$

$$= 1 - (x-1) - \frac{1^2}{2} + \frac{(x-1)^2}{2}$$

$$g(x) = 1 - x + 1 - \frac{1}{2} + \frac{(x^2 - 2x + 1)}{2}$$

$$= 1 - x + 1 - \frac{1}{2} + \frac{1}{2}x^2 - x + \frac{1}{2}$$

$$= \frac{1}{2}x^2 - 2x + 2$$

$$g(x) = \frac{1}{2}x^2 - 2x + 2 \quad \text{when} \quad 1 < x < 2$$

So

$$g(x) = \begin{cases} 0 & ; \ x < -2 \\ \frac{1}{2}x^2 + 2x + 2 & ; \ -2 < x < -1 \\ 1 - \frac{1}{2}x^2 & ; \ -1 < x < 1 \\ \frac{1}{2}x^2 - 2x + 2 & ; \ 1 < x < 2 \\ 0 & ; \ x > 2 \end{cases}$$