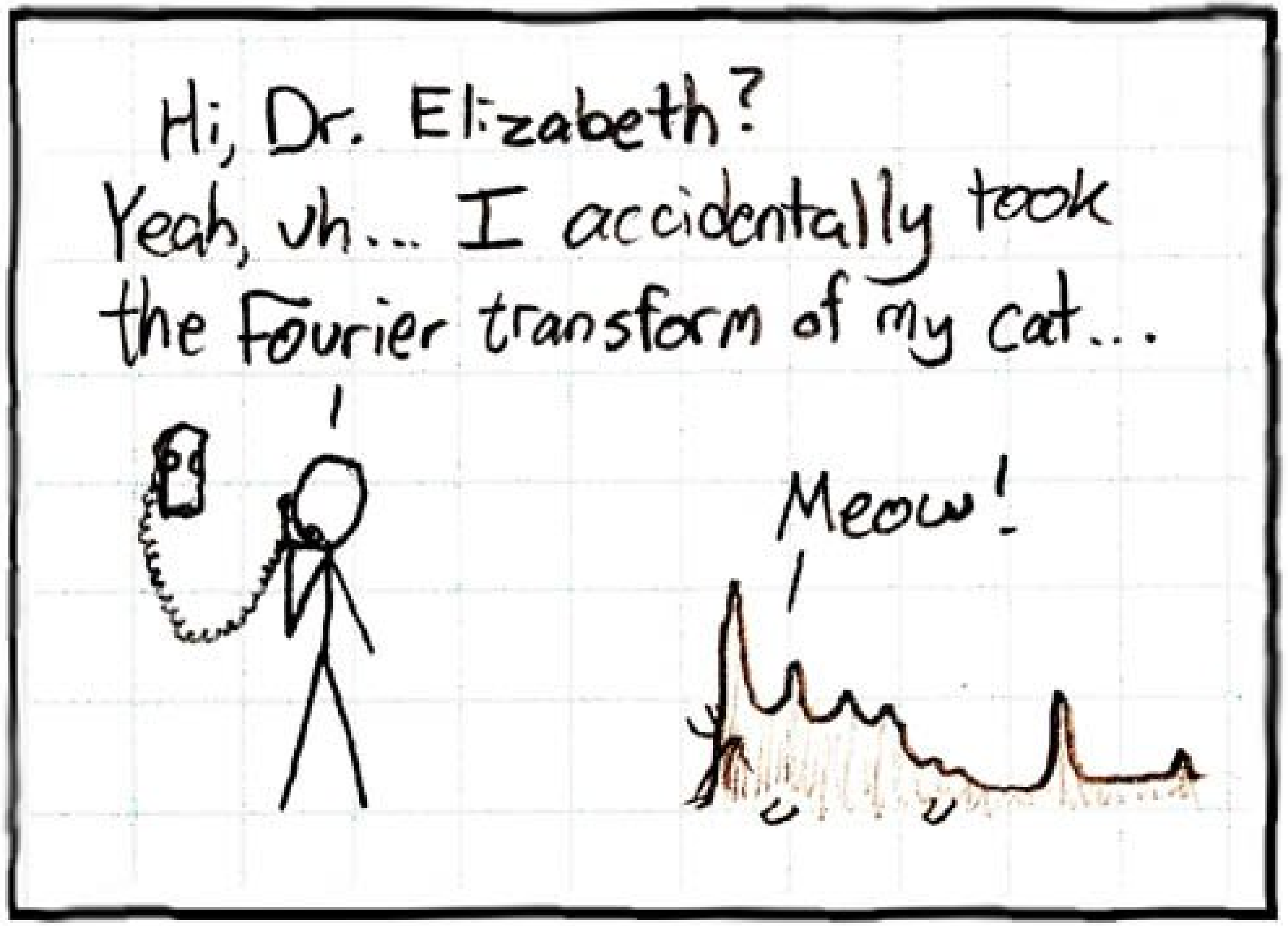
Fourier Transform



Introduction to Computer Vision
Spring 2026, Lecture 2.3

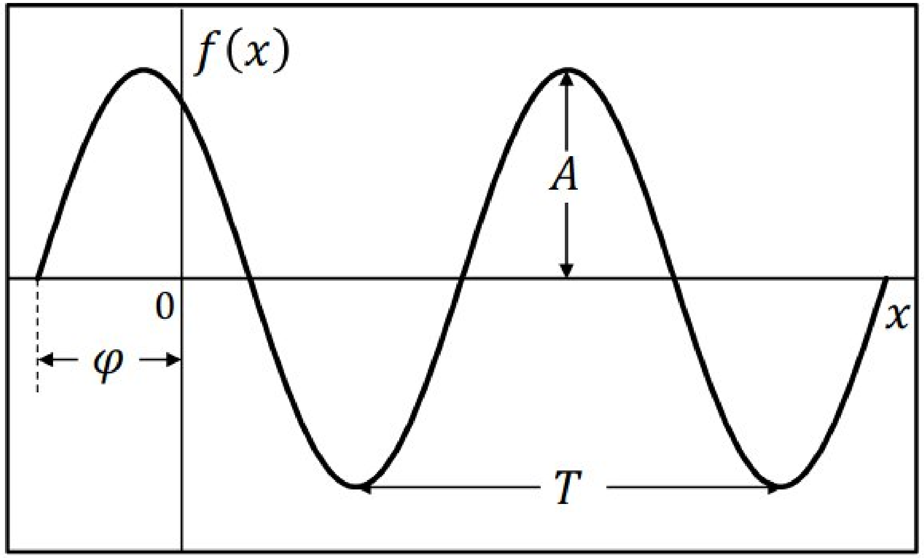
https://liaoux.github.io/uir-cv-2026/

Jean Baptiste Joseph Fourier

Any Periodic Function can be rewritten as a Weighted Sum of Infinite Sinusoids of Different Frequencies.

Sinusoid

$$f(x) = A\sin(2\pi ux + \varphi)$$



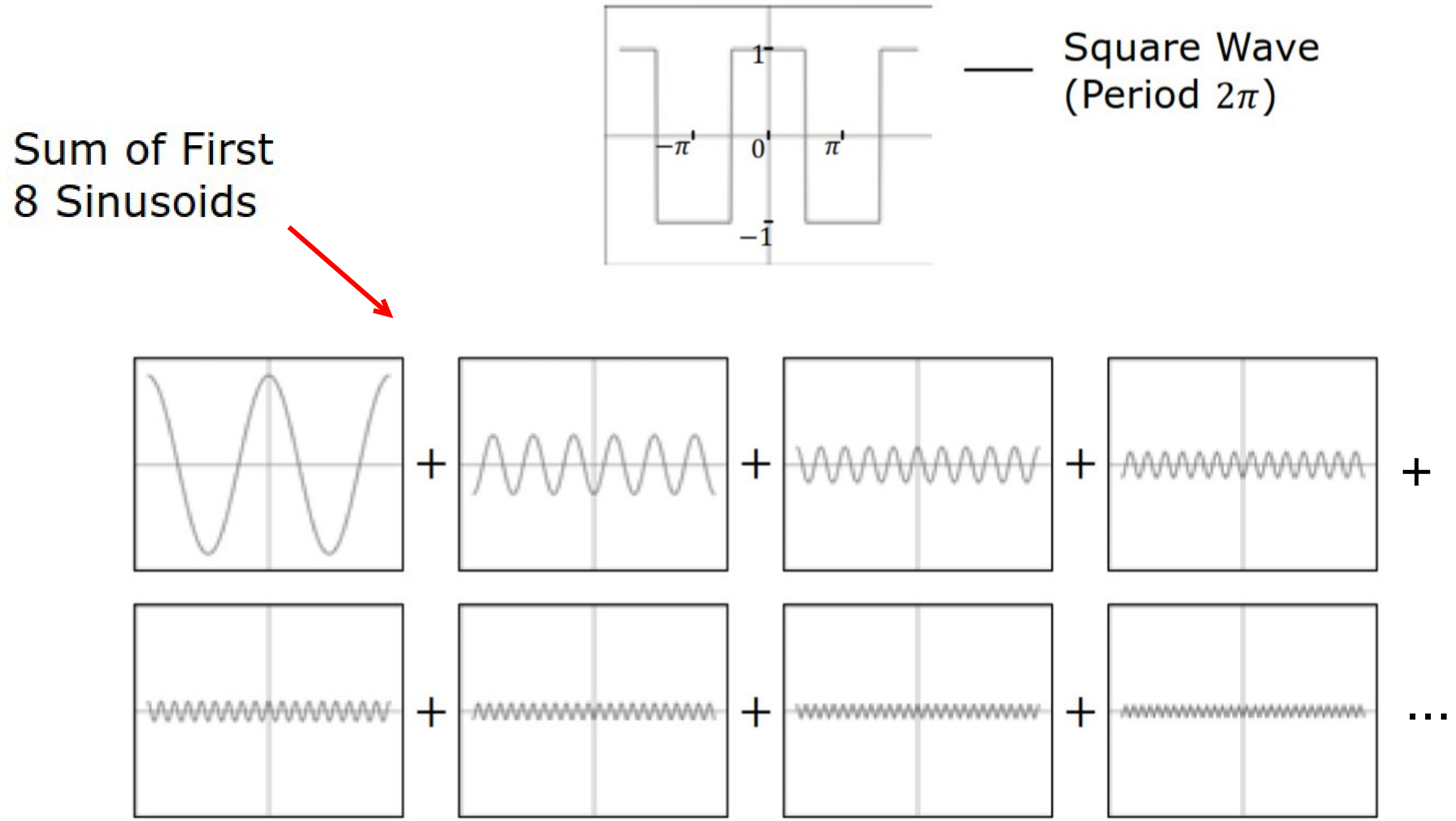
A: Amplitude

φ: Phase

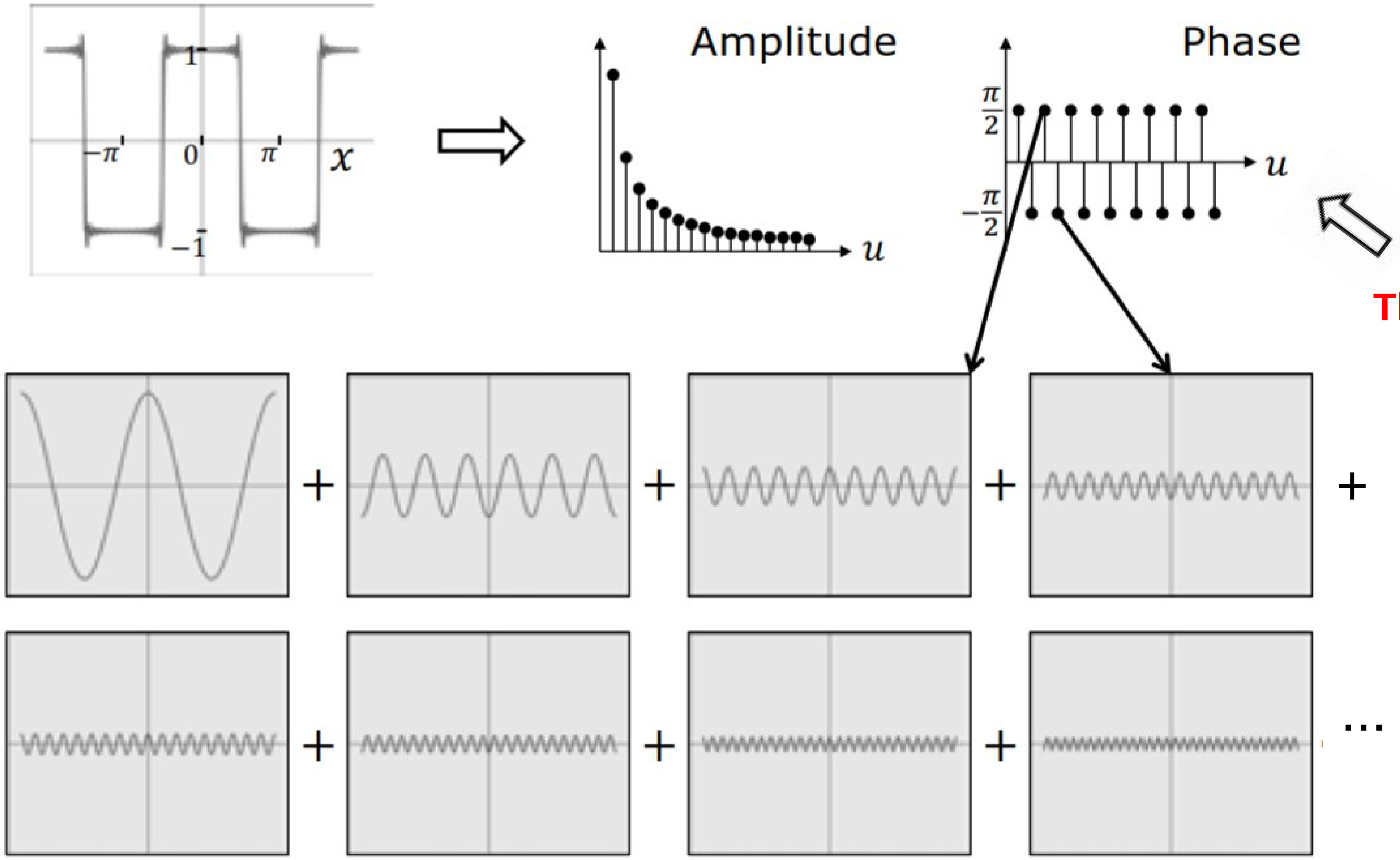
T: Period

u: Frequency $(1/T)$

Fourier Series

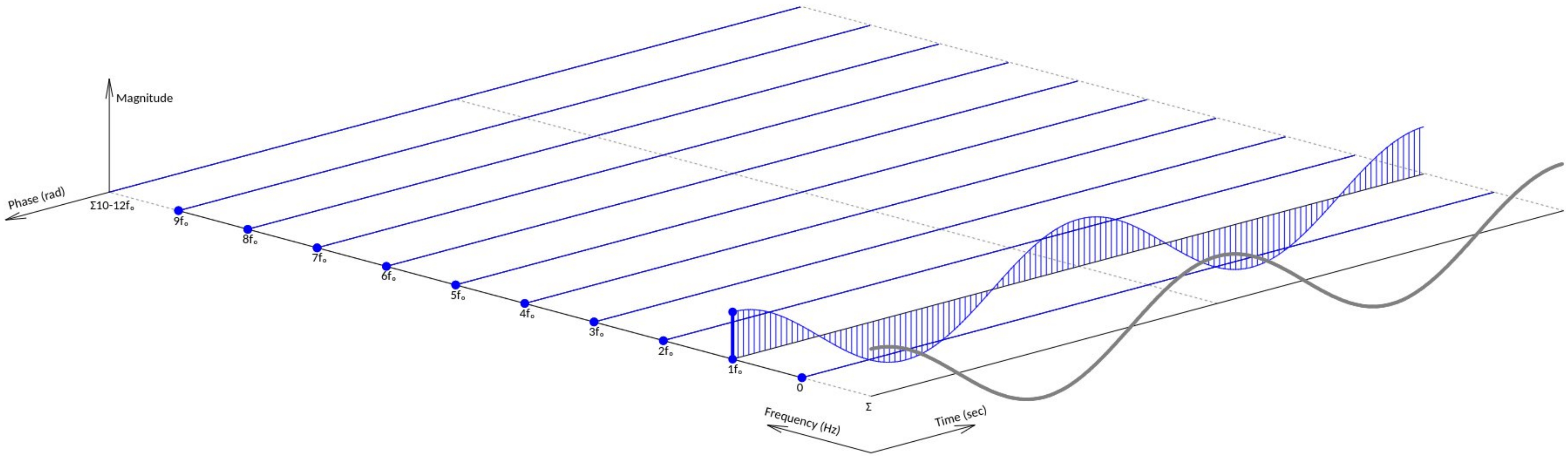


Frequency Representation of Signal



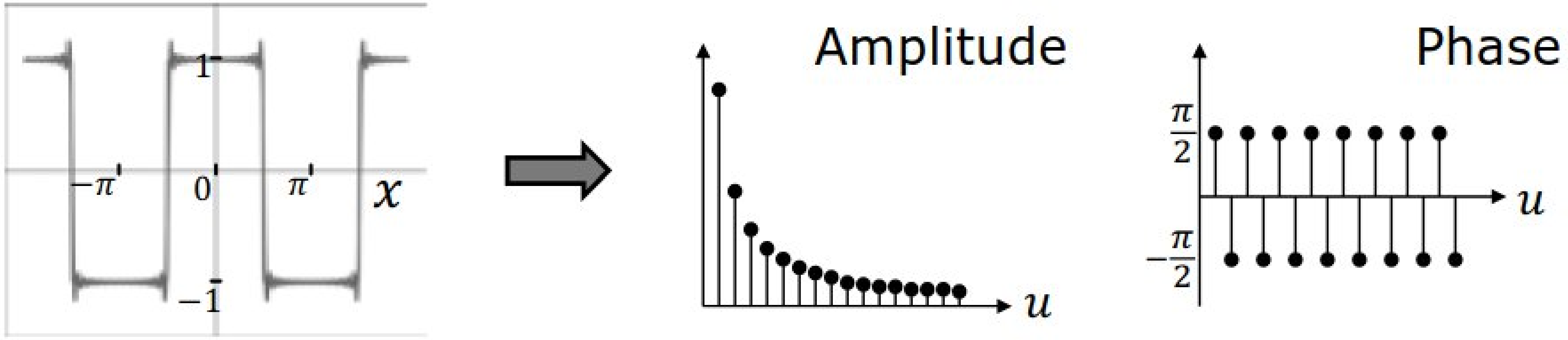
The Frequency domain

Fourier Transform Online Visualizer



https://tomasboril.cz/fourierseries3d/en/

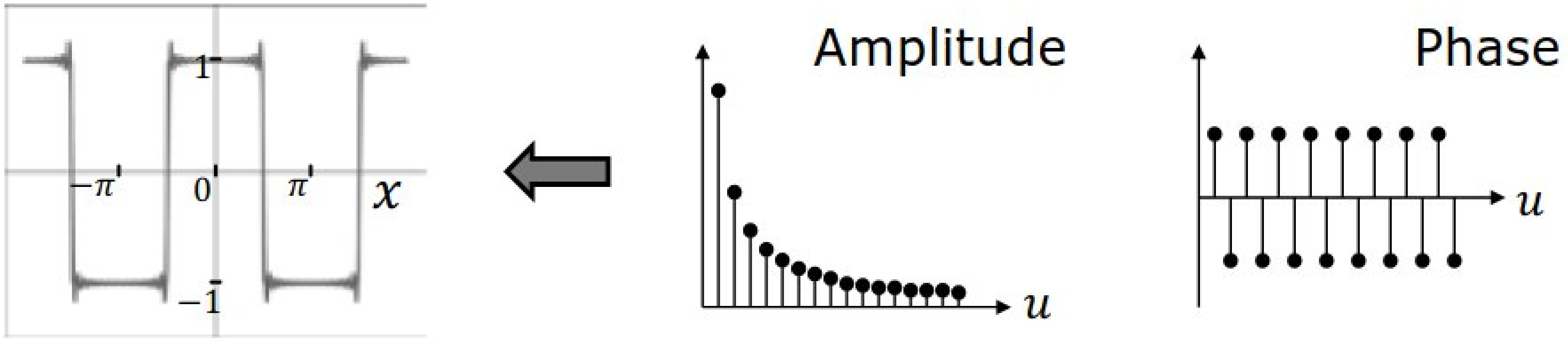
Fourier Transform (FT)



Represents a signal **f**(**x**) in terms of Amplitudes and Phases of its Constituent Sinusoids.

$$f(x) \longrightarrow \boxed{\mathrm{FT}} \longrightarrow F(u)$$

Inverse Fourier Transform (IFT)



Computes the signal $\mathbf{f}(\mathbf{x})$ from the Amplitudes and Phases of its Constituent Sinusoids.

$$f(x) \longleftarrow \boxed{\text{IFT}} \longleftarrow F(u)$$

Finding FT and IFT

Fourier Transform:

$$\boxed{F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx}$$

x: space

u: frequency

$e^{i\theta} = \cos\theta + i\sin\theta$

$i = \sqrt{-1}$

Inverse Fourier Transform:

$$\boxed{f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ux} du}$$

Finding FT and IFT

Fourier Transform:

$$\boxed{F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx}$$

x: space

u: frequency

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$i = \sqrt{-1}$$

Inverse Fourier Transform:

$$\boxed{f(x) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ux} du}$$

Complex Exponential (Euler Formula)

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \qquad \boxed{i = \sqrt{-1}}$$

Expand $e^{i\theta}$ using Taylor Series:

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \ldots$$

Complex Exponential (Euler Formula)

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \qquad \boxed{i = \sqrt{-1}}$$

Expand $e^{i\theta}$ using Taylor Series:

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \ldots$$

Then:

$$e^{i\theta} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)}_{\sin\theta}$$

even powers

odd powers

Fourier Transform is Complex!

$F(u)$ holds the Amplitude and Phase of the sinusoid of frequency u.

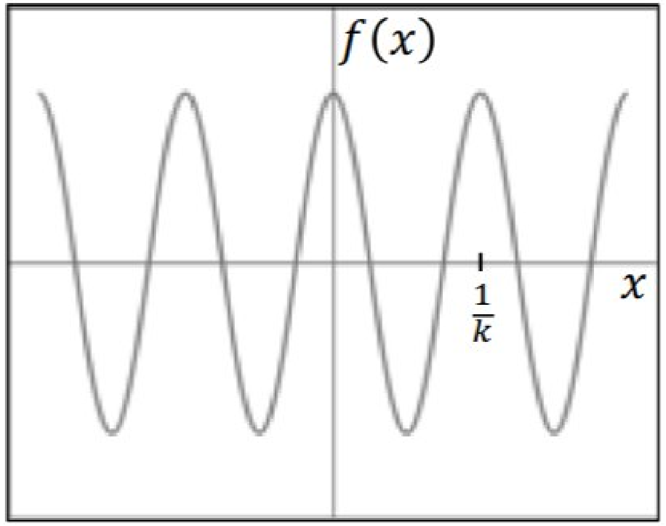
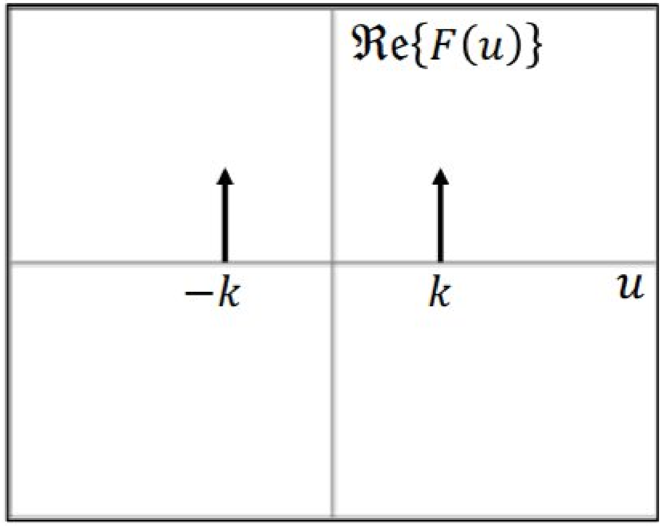
$$\boxed{F(u) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi ux} dx}$$

$$F(u) = \Re\mathrm{e}\{F(u)\} + i\,\Im\mathrm{m}\{F(u)\}$$

Amplitude: $A(u) = \sqrt{\Re\mathrm{e}\{F(u)\}^2 + \Im\mathrm{m}\{F(u)\}^2}$

Phase: $\varphi(u) = \mathrm{atan2}(\Im\mathrm{m}\{F(u)\}, \Re\mathrm{e}\{F(u)\})$

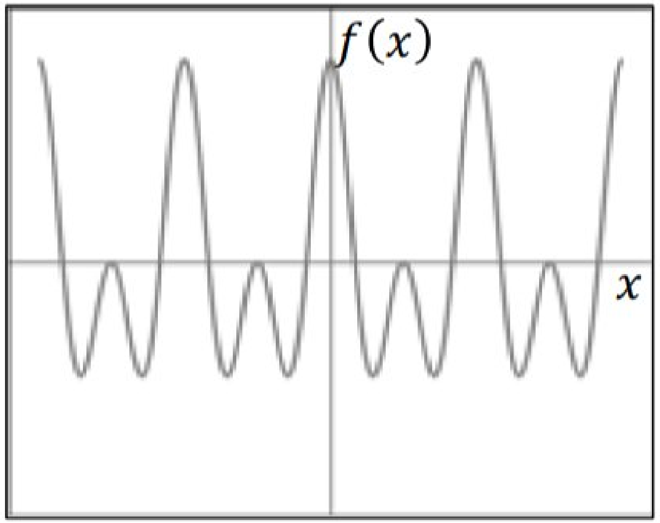
Fourier Transform Examples

Signal $f(x)$	Fourier Transform $F(u)$
	
$f(x) = \cos 2\pi kx$	$F(u) = \frac{1}{2}[\delta(u+k) + \delta(u-k)]$

The Real part of the spectrum

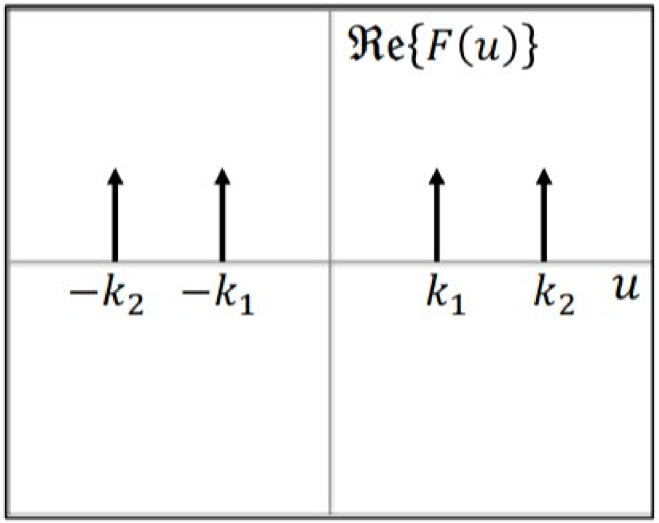
Fourier Transform Examples

Signal $f(x)$



$$f(x) = \cos 2\pi k_1 x + \cos 2\pi k_2 x$$

Fourier Transform $F(u)$

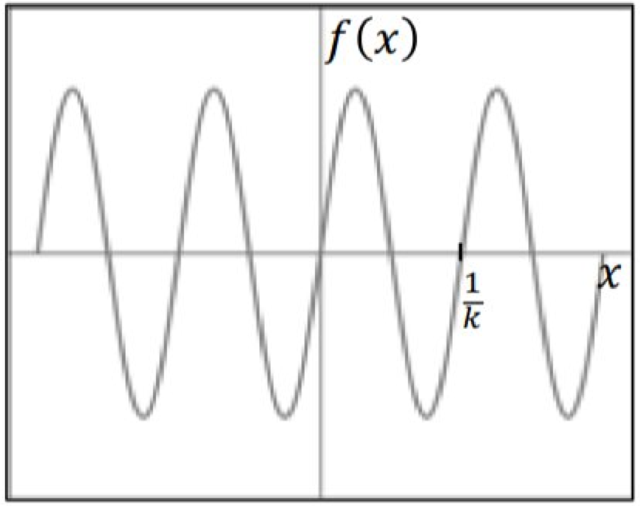


$$F(u) = \frac{1}{2}\left[\delta(u + k_1) + \delta(u - k_1) + \delta(u + k_2) + \delta(u - k_2)\right]$$

The Real part of the spectrum

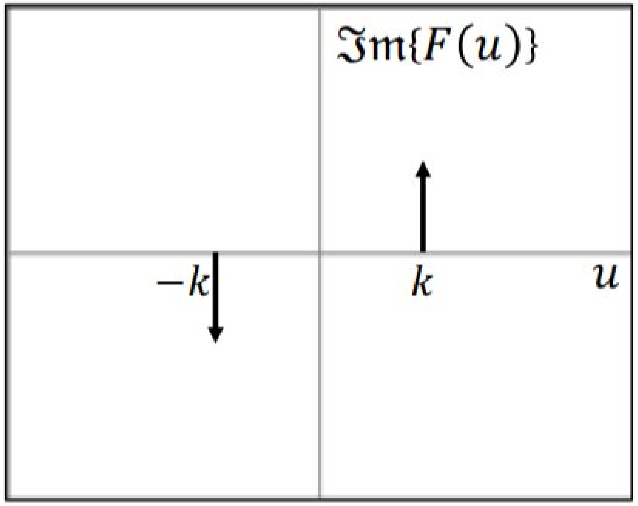
Fourier Transform Examples

Signal $f(x)$



$$f(x) = \sin 2\pi kx$$

Fourier Transform $F(u)$



The Imaginary part of the spectrum

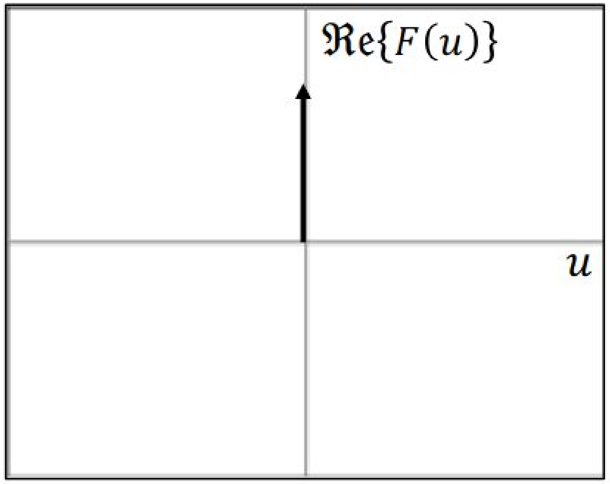
$$F(u) = \frac{1}{2}i[\delta(u+k) - \delta(u-k)]$$

Fourier Transform Examples

Signal $f(x)$

$$f(x) = 1$$

Fourier Transform $F(u)$

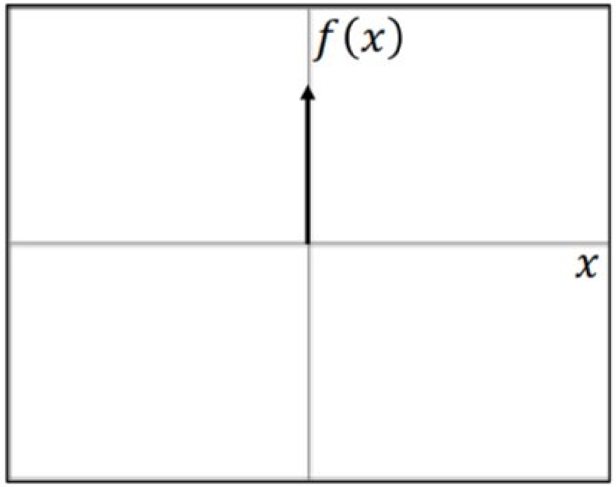


The Real part of the spectrum

$$F(u) = \delta(u)$$

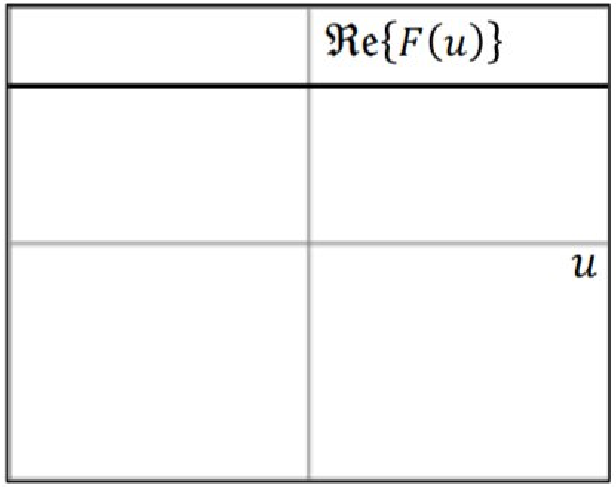
Fourier Transform Examples

Signal $f(x)$



$$f(x) = \delta(x)$$

Fourier Transform $F(u)$

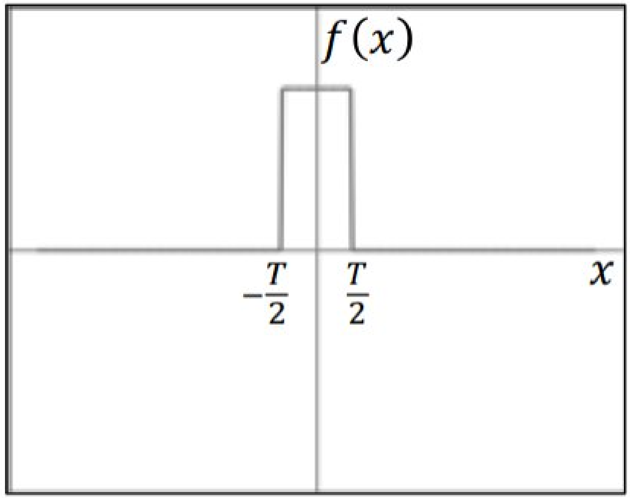


The Real part of the spectrum

$$F(u) = 1$$

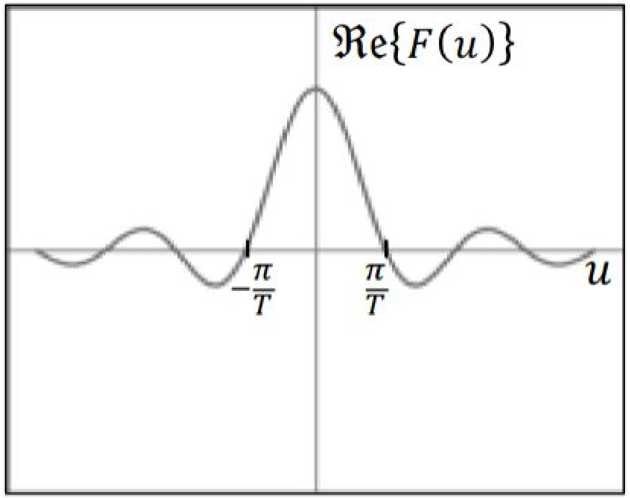
Fourier Transform Examples

Signal $f(x)$



$$f(x) = \mathrm{Rect}\left(\frac{x}{T}\right)$$

Fourier Transform $F(u)$

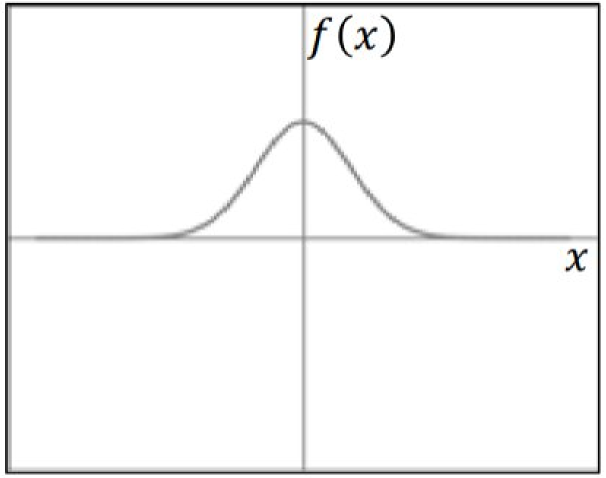


The Real part of the spectrum

$$F(u) = T\,\mathrm{sinc}\,Tu$$

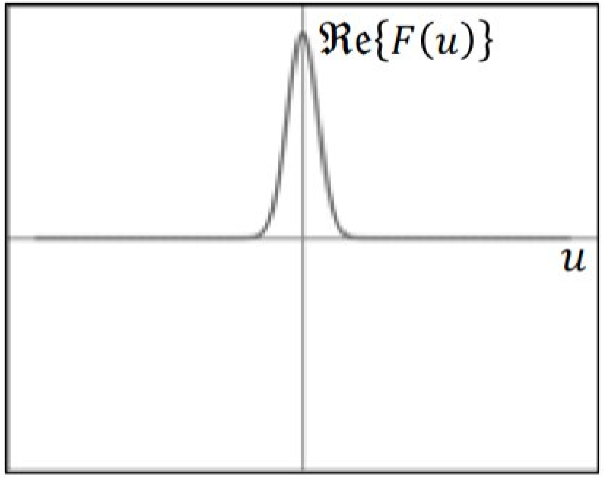
Fourier Transform Examples

Signal $f(x)$



$$f(x) = e^{-ax^2}$$

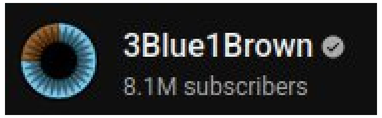
Fourier Transform $F(u)$

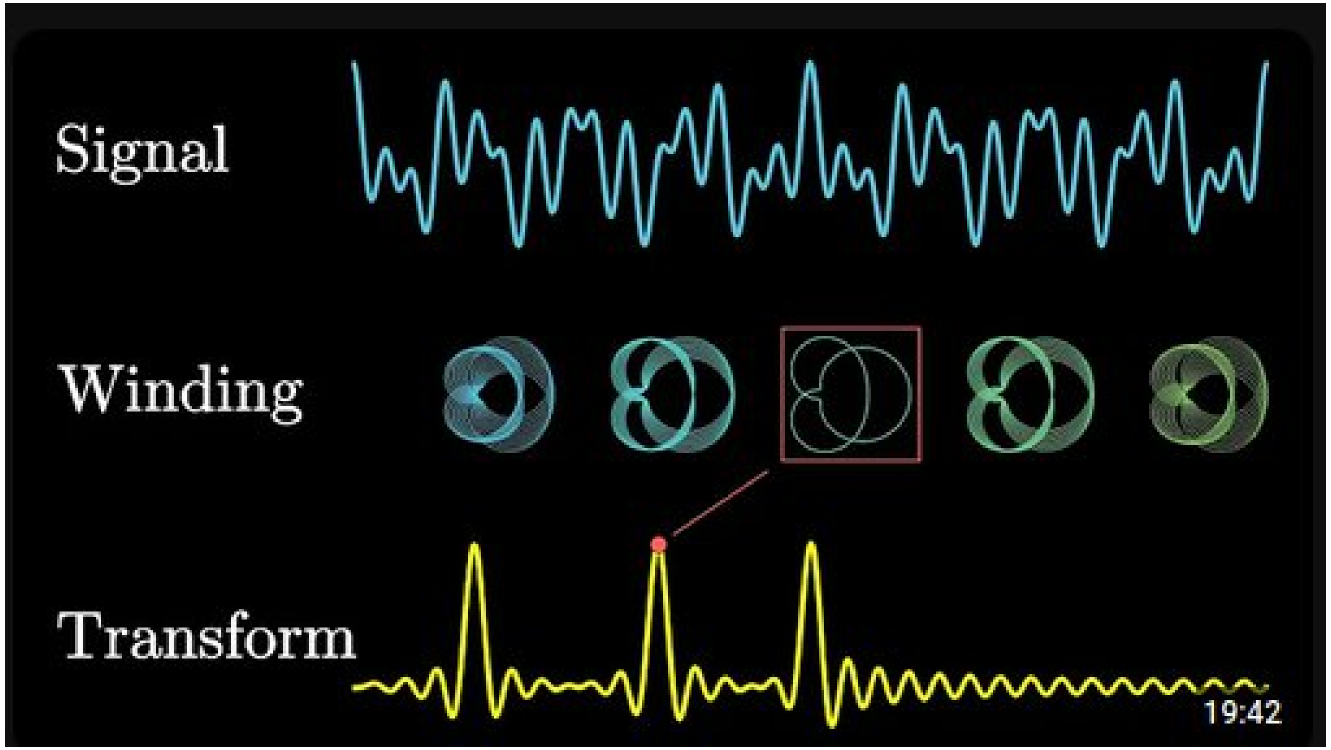


The Real part of the spectrum

$$F(u) = \sqrt{\pi/a}\; e^{-\pi^2 u^2/a}$$

But what is the Fourier Transform? A visual introduction





https://youtu.be/spUNpyF58BY?si=pW-qMWyXnc_kQBfQ

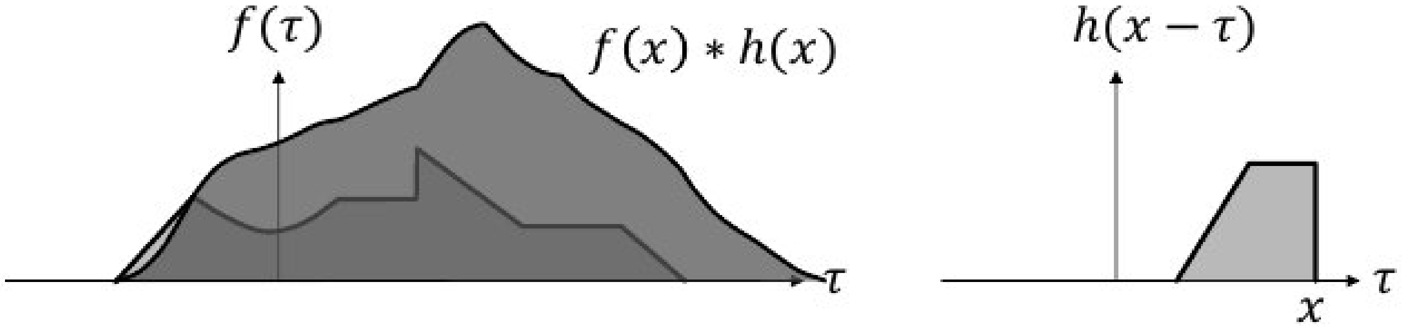
Properties of Fourier Transform

Property	Spatial Domain	Frequency Domain
Linearity	$\alpha f_1(x) + \beta f_2(x)$	$\alpha F_1(u) + \beta F_2(u)$
Scaling	$f(ax)$	$\frac{1}{\lvert a \rvert} F\left(\frac{u}{a}\right)$
Shifting	$f(x-a)$	$e^{-i2\pi ua} F(u)$
Differentiation	$\frac{d^n}{dx^n}\left(f(x)\right)$	$(i2\pi u)^n F(u)$

Recall: The Convolution

The convolution of two functions $f(x)$ and $h(x)$ is:

$$\boxed{g(x) = f(x) * h(x) = \int_{-\infty}^{\infty} f(\tau)h(x-\tau)\,d\tau}$$



LSIS implies Convolution and Convolution implies LSIS

The Convolution Theorem

Spatial Domain		Frequency Domain
$g(x) = f(x) * h(x)$ Convolution	$\longleftrightarrow$	$G(u) = F(u)\,H(u)$ Multiplication
$g(x) = f(x)\,h(x)$ Multiplication	$\longleftrightarrow$	$G(u) = F(u) * H(u)$ Convolution

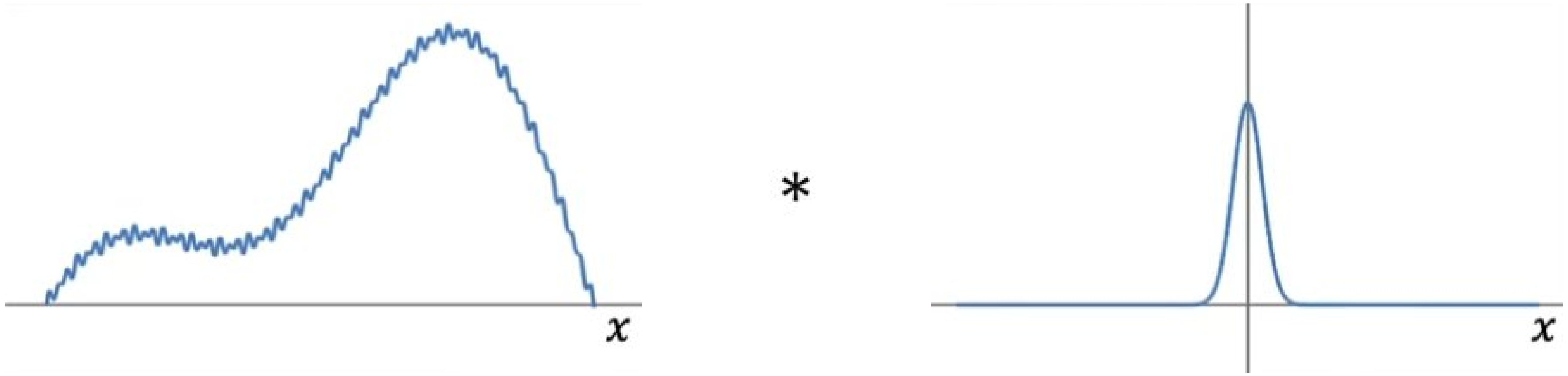
Convolution and Fourier Transform

The Spatial domain

$$\begin{tikzcd} g(x) & = & f(x) \arrow[d, "\boxed{\mathrm{FT}}"] & * & h(x) \arrow[d, "\boxed{\mathrm{FT}}"] \\ G(u) \arrow[u, "\boxed{\mathrm{IFT}}"] & = & F(u) & \times & H(u) \end{tikzcd}$$

The frequency domain

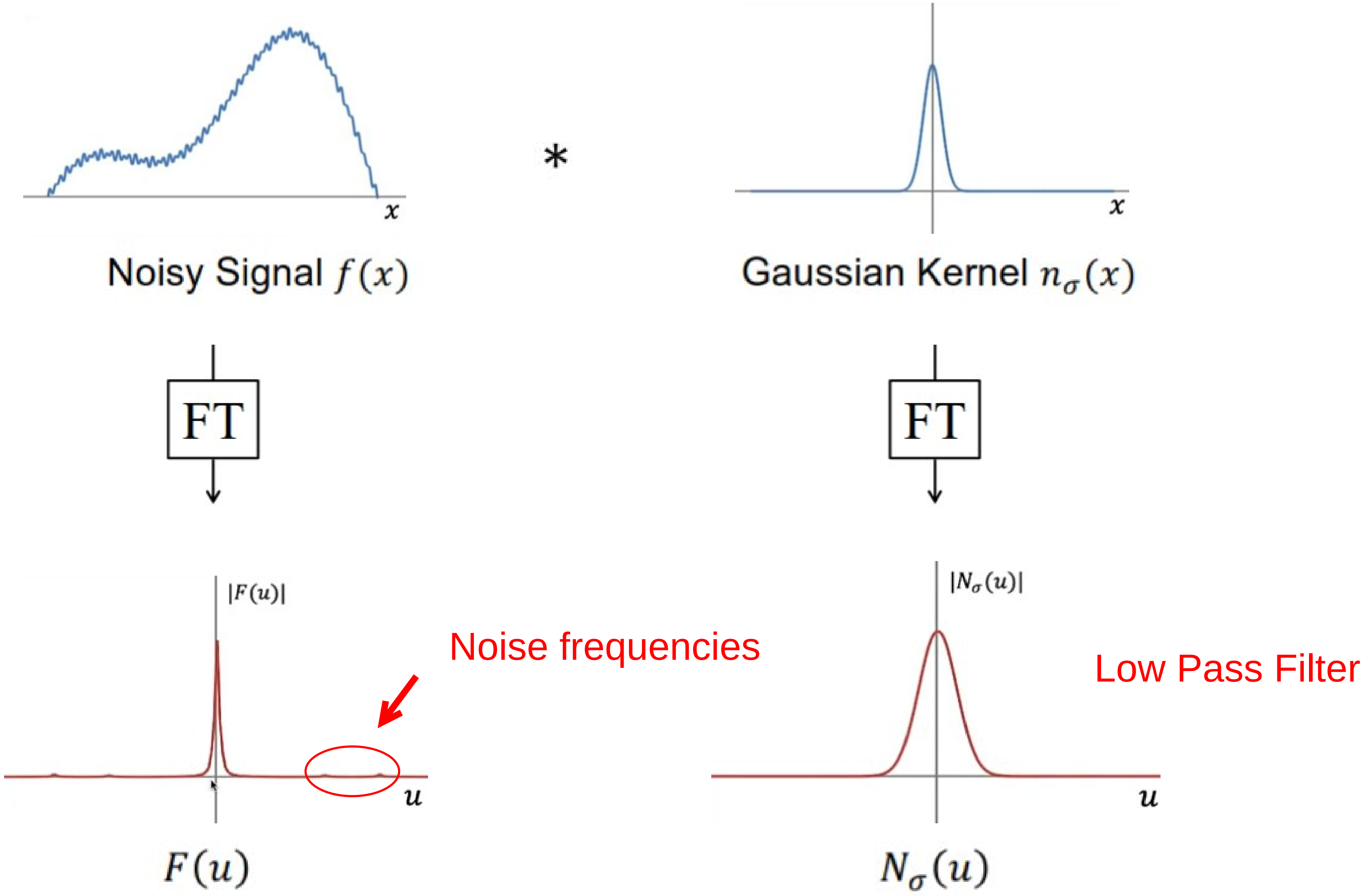
Gaussian Smoothing in Fourier Domain



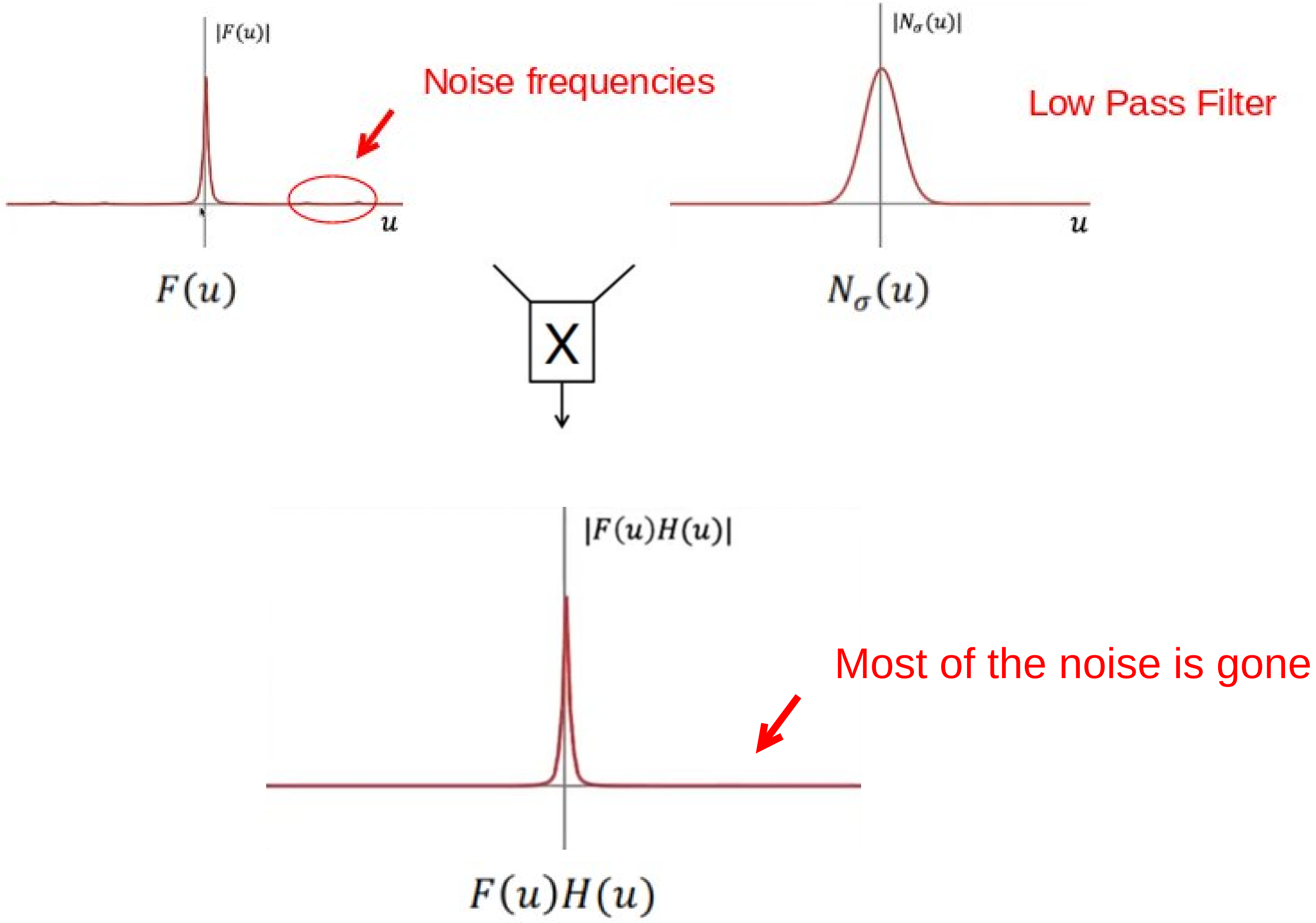
Noisy Signal $f(x)$ Gaussian Kernel $n_\sigma(x)$

Convolve the Noisy Signal with a Gaussian Kernel

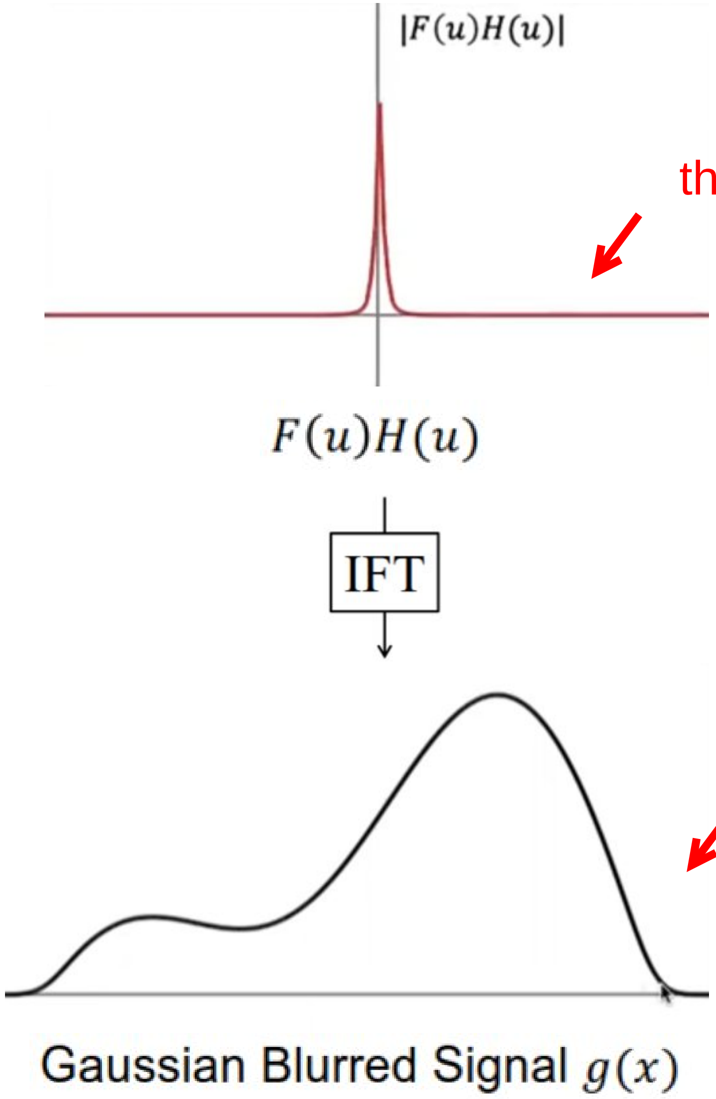
Gaussian Smoothing in Fourier Domain



Gaussian Smoothing in Fourier Domain



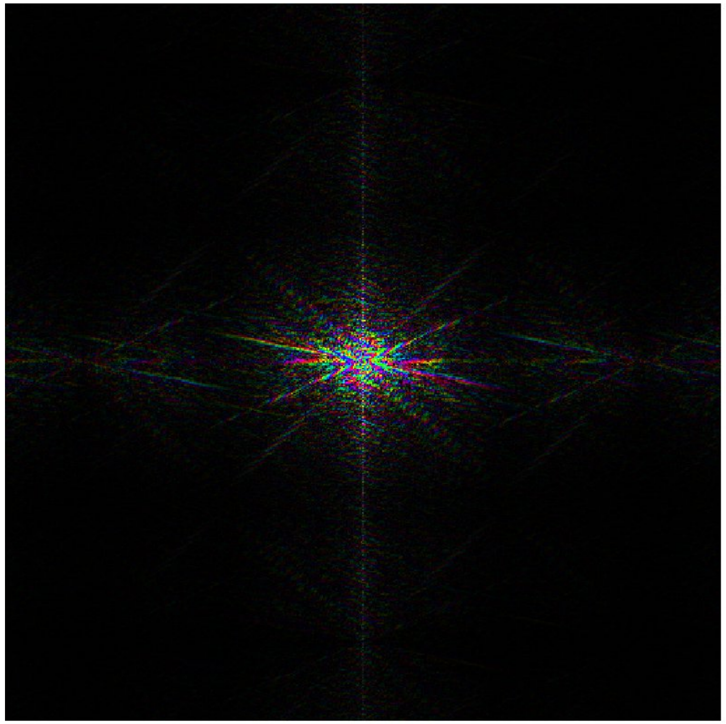
Gaussian Smoothing in Fourier Domain



the noise is gone !

a clean filtered signal !

Image Filtering in the Frequency domain



https://liaoux.github.io/uir-cv-2026/

Introduction to Computer Vision
Spring 2026, Lecture 2.4

2D Fourier Transform: Discrete Images

Discrete Fourier Transform (DFT):

$$\boxed{F[p,q] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-i2\pi pm/M} e^{-i2\pi qn/N}}$$

$p = 0 \ldots M-1$

$q = 0 \ldots N-1$

p and q are frequencies along m and n, respectively

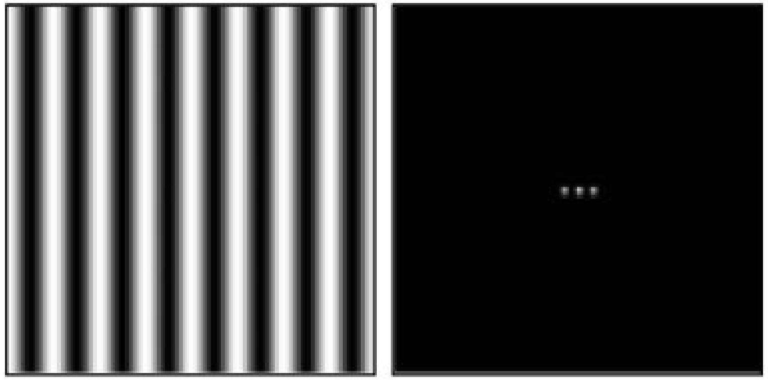
Inverse Discrete Fourier Transform (IDFT):

$$\boxed{f[m,n] = \frac{1}{MN} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} F[p,q] e^{i2\pi pm/M} e^{i2\pi qn/N}}$$

$m = 0 \ldots M-1$

$n = 0 \ldots N-1$

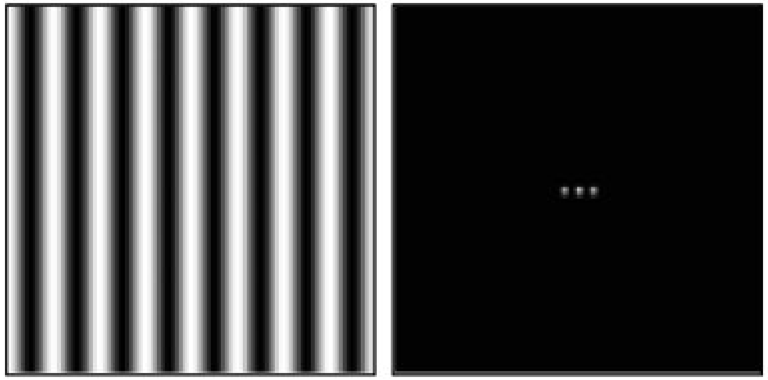
2D Fourier Transform: Simple images



$f(m,n)$ $\log(|F(p,q)|)$

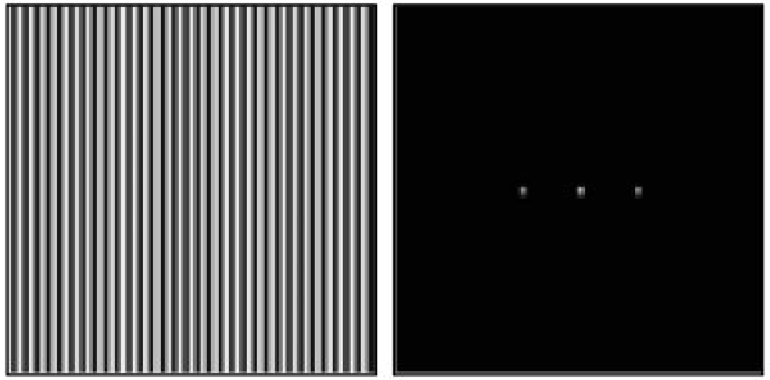
cosine function

2D Fourier Transform: Simple images



$f(m,n)$ $\log(|F(p,q)|)$

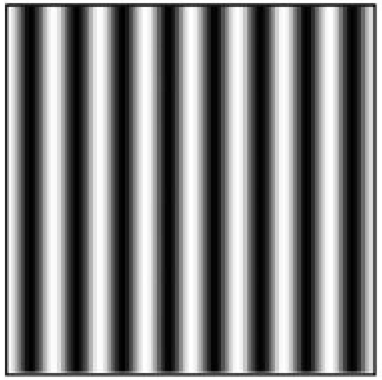
cosine function



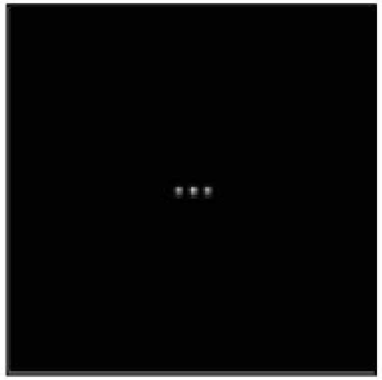
$g(m,n)$ $\log(|G(p,q)|)$

cosine function
with higher frequency

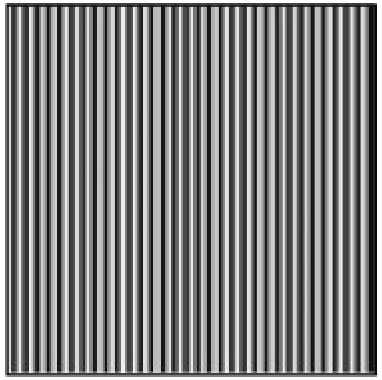
2D Fourier Transform: Simple images



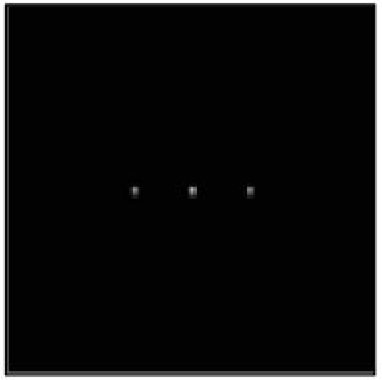
$f(m,n)$



$\log(|F(p,q)|)$



$g(m,n)$

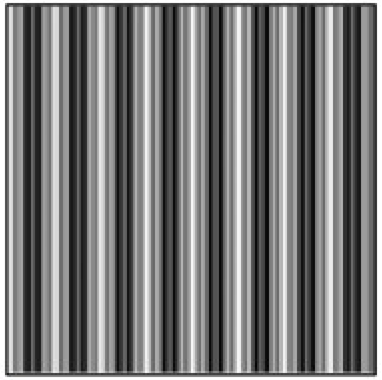


$\log(|G(p,q)|)$

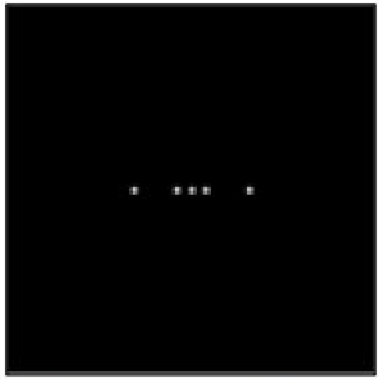
cosine function

cosine function
with higher frequency

Sum of two cosine
functions



$f(m,n) + g(m,n)$

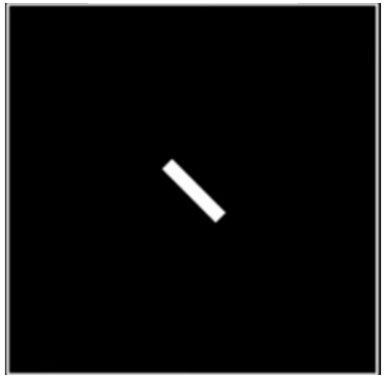


$\log(|F(p,q) + G(p,q)|)$

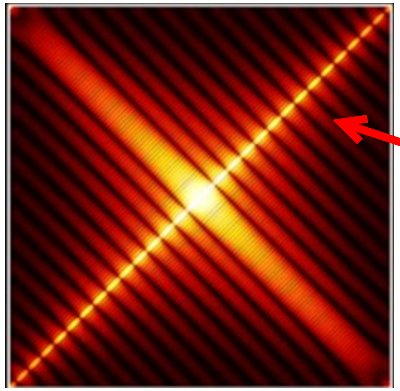
2D Fourier Transform: Binary images

$f(m,n)$ $\log(|F(p,q)|)$

Line



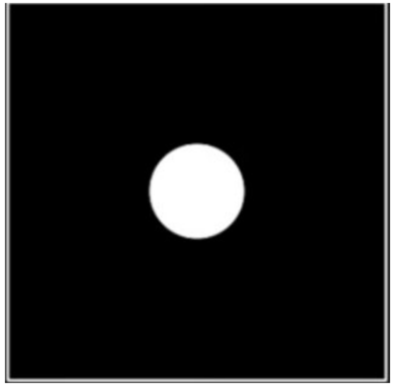
FT



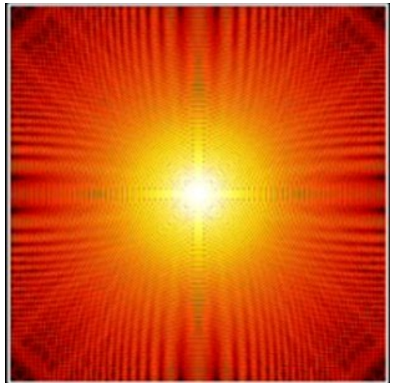
Edge frequencies

Recall: The further you get from the center, the higher the frequencies become.

Circle



FT



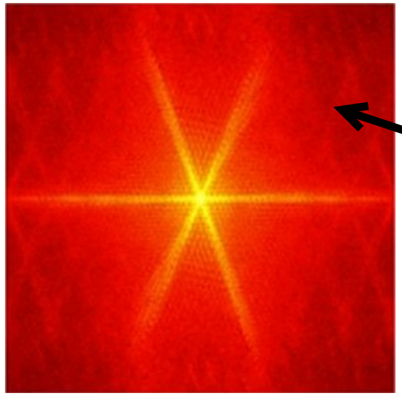
Min  Max

2D Fourier Transform: Natural images

$f(m,n)$ $\quad\quad$ $\log(|F(p,q)|)$

Rubik's cube

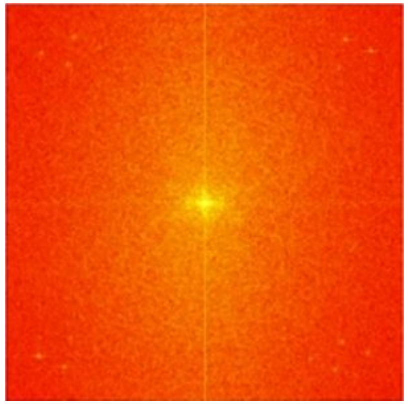
FT



Edge frequencies

Mandrill

FT



Recall: The further you get from the center, the higher the frequencies become.

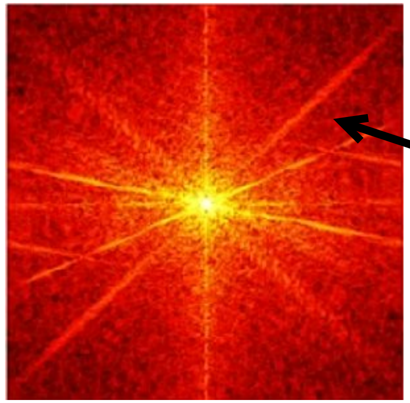
Min  Max

2D Fourier Transform: Complex images

$f(m,n)$ $\log(|F(p,q)|)$

Camera man

FT



Edge frequencies

Noise

FT

Recall: The further you get from the center, the higher the frequencies become.

Min  Max

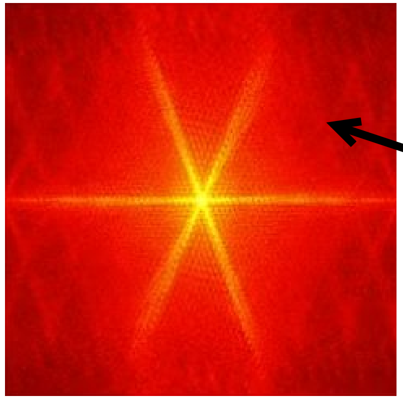
Low Pass Filtering

$f(m,n)$

Rubik's cube

FT

$\log(|F(p,q)|)$

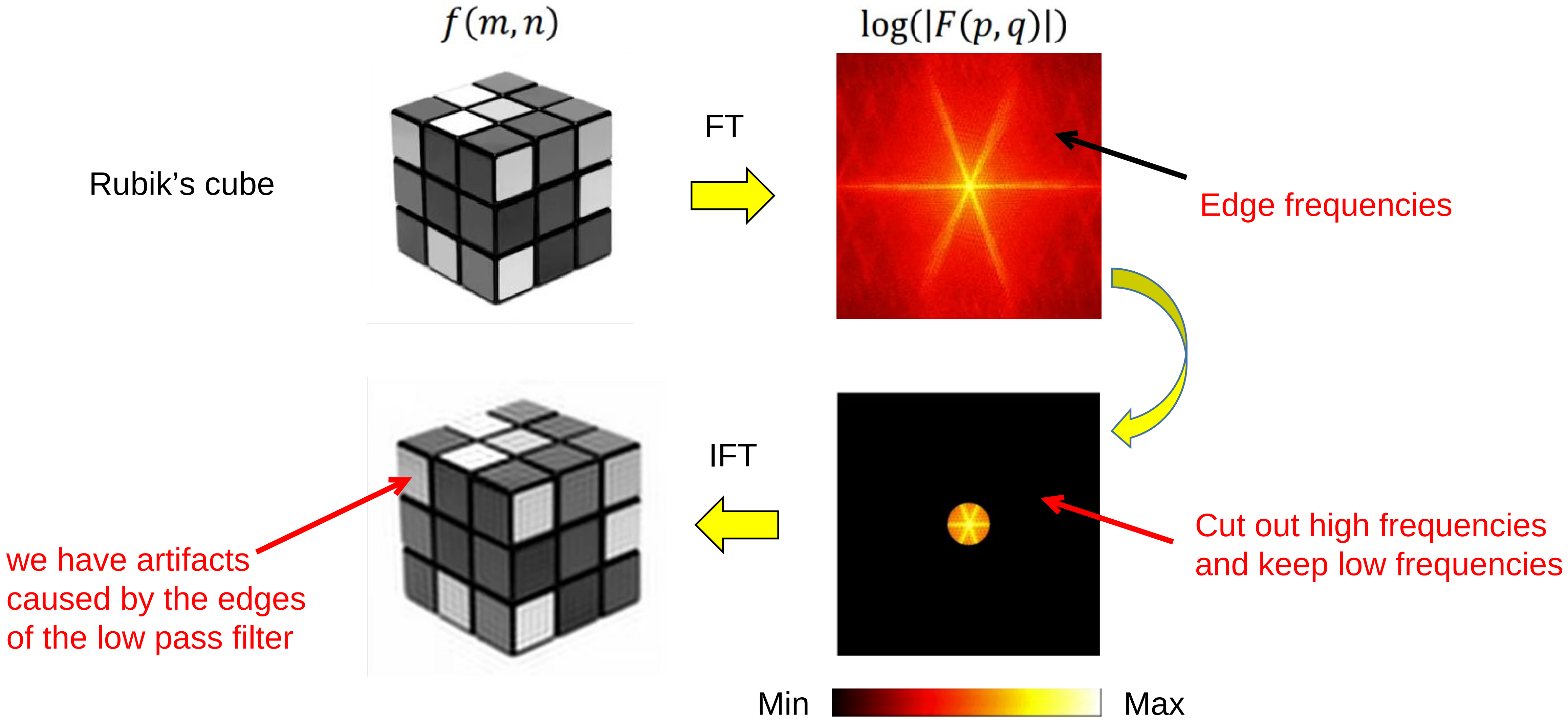


Edge frequencies

Recall: The further you get from the center, the higher the frequencies become.

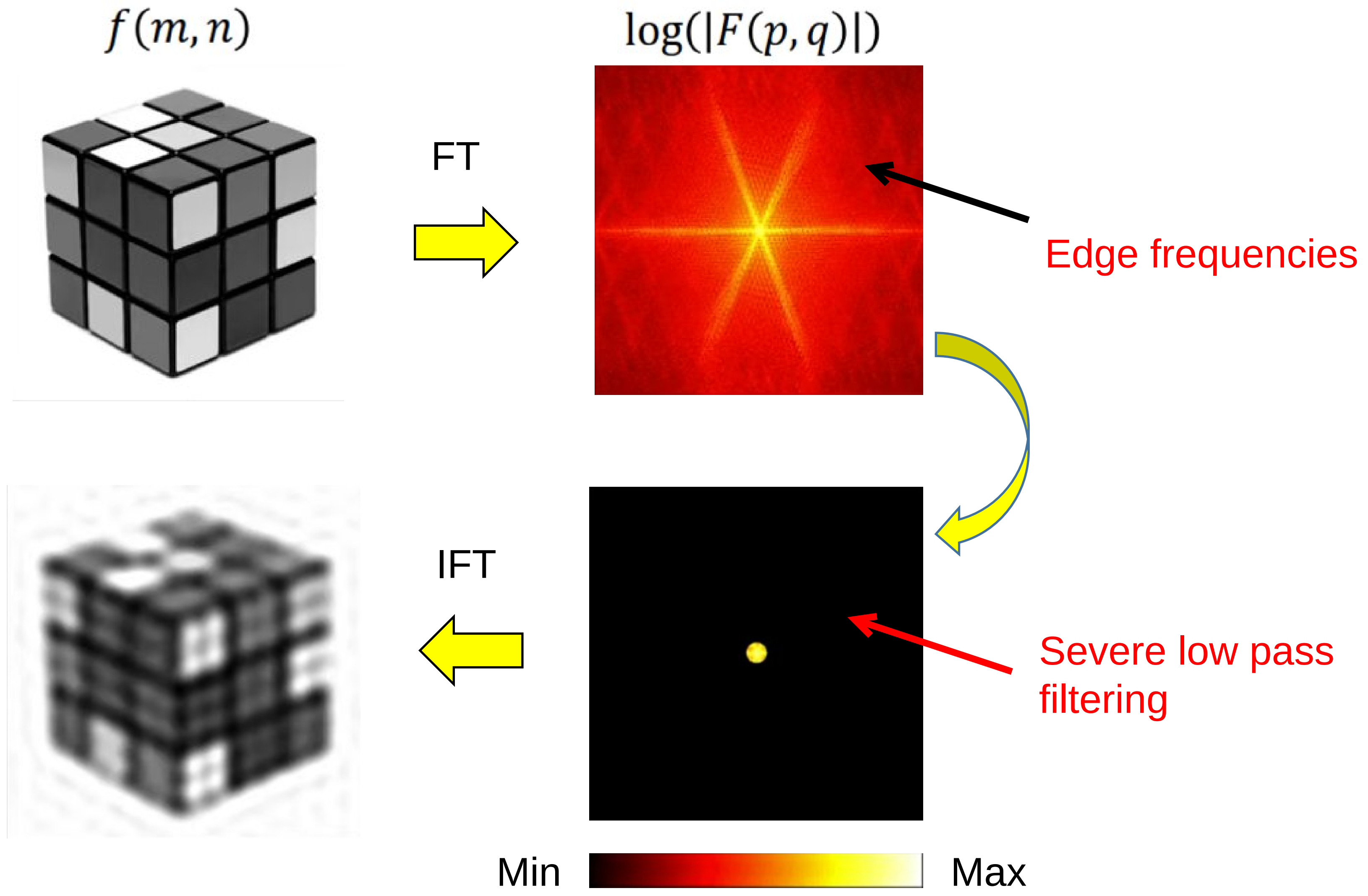
Min Max

Low Pass Filtering

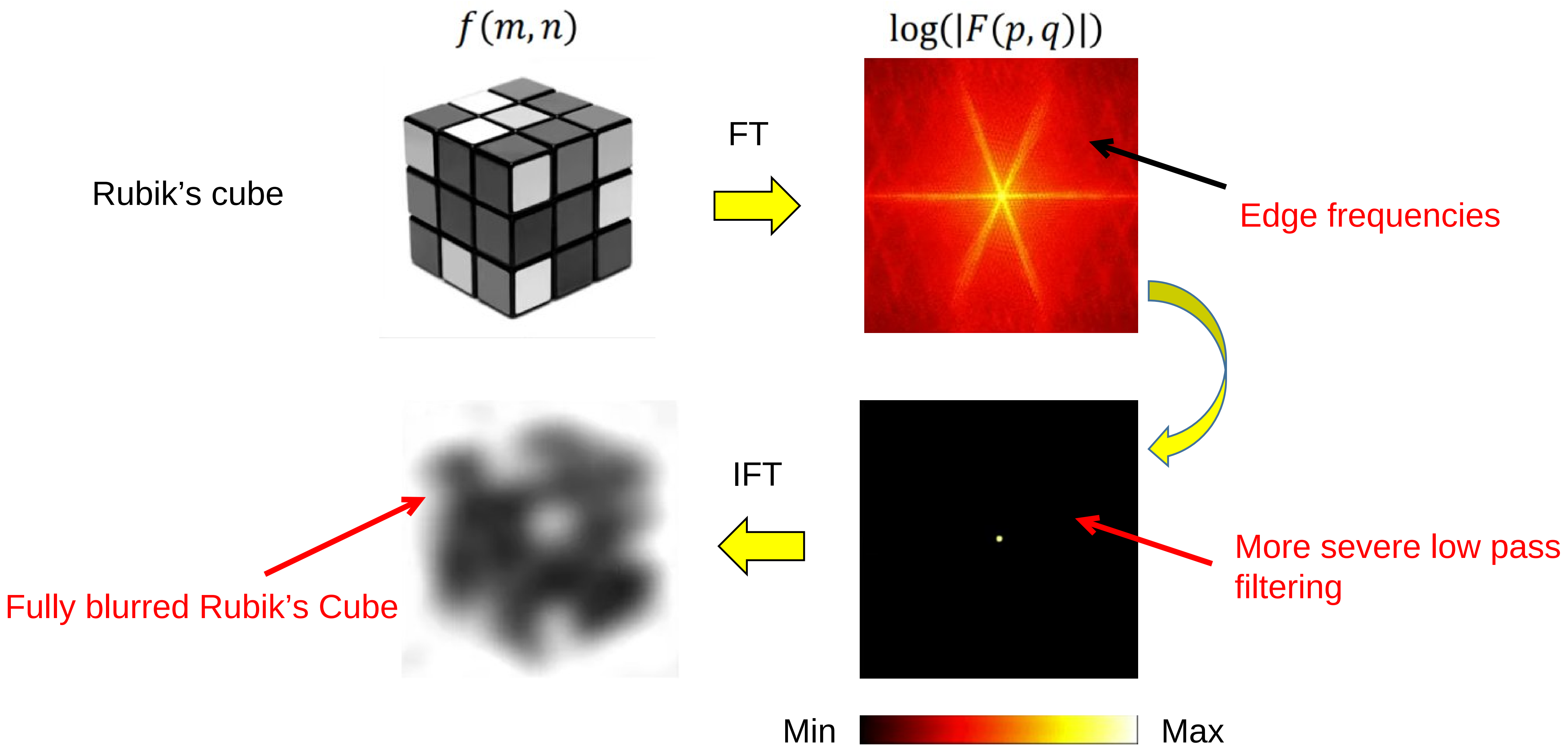


Severe Low Pass Filtering

Rubik's cube



More Severe Low Pass Filtering

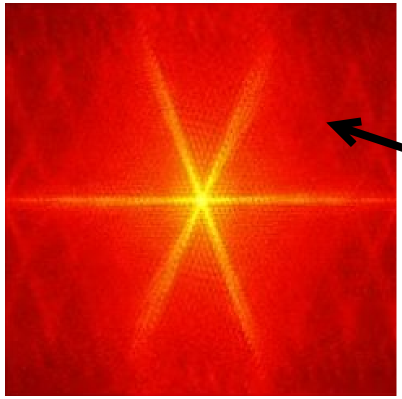


High Pass Filtering

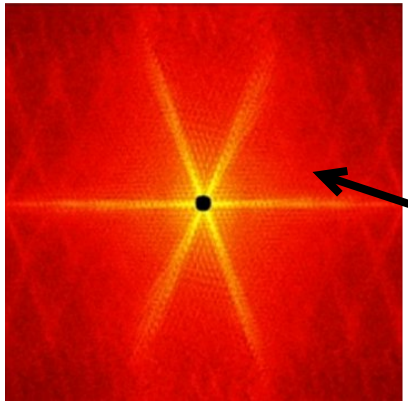
$f(m,n)$ $\log(|F(p,q)|)$

Rubik's cube

FT

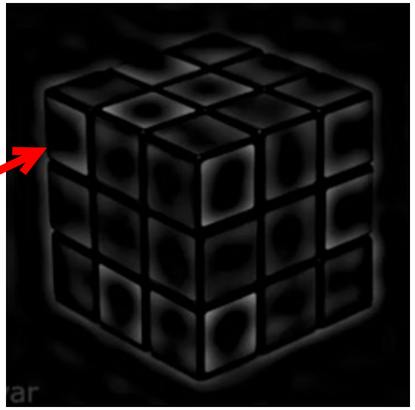


Edge frequencies



Cut out low frequencies
and keep high frequencies

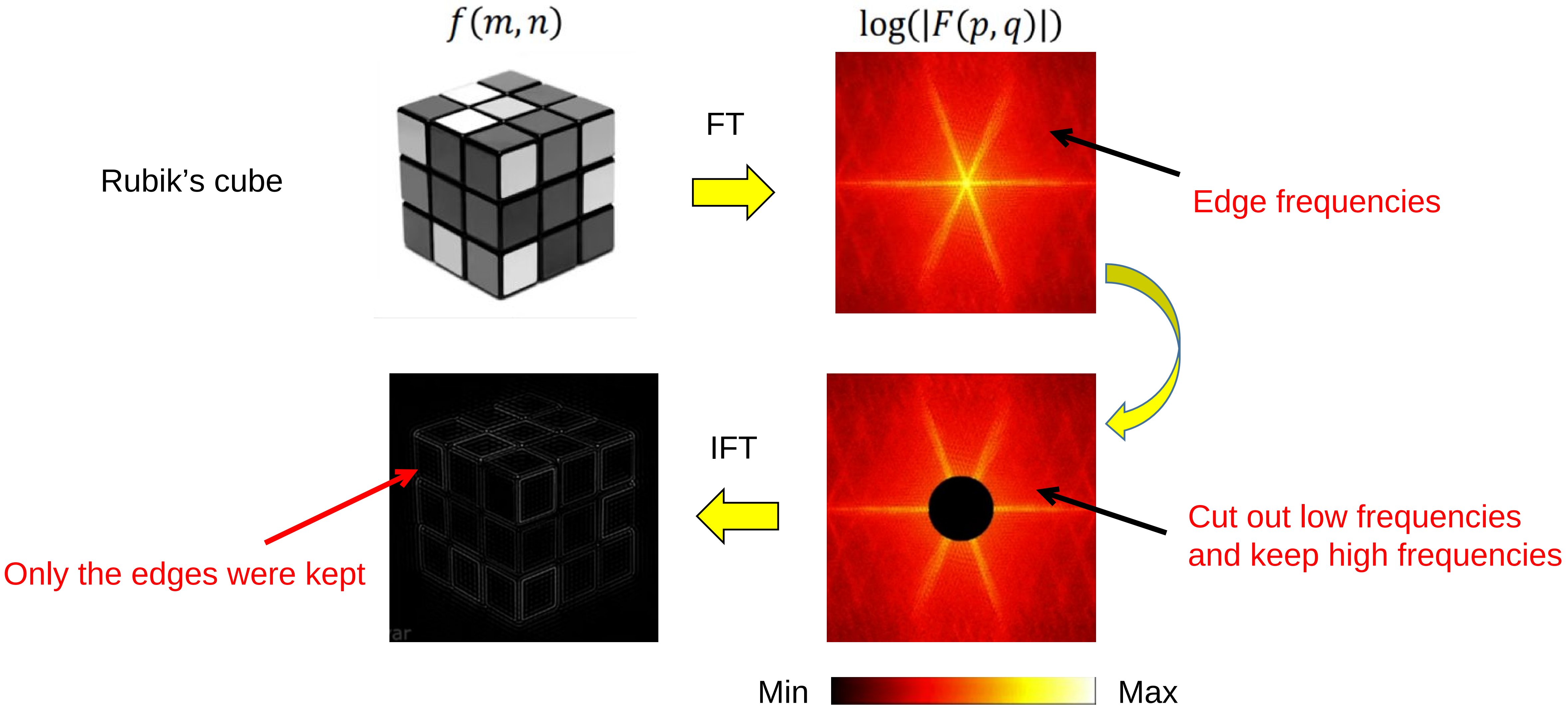
IFT



Edges begin to appear

Min  Max

Severe High Pass Filtering



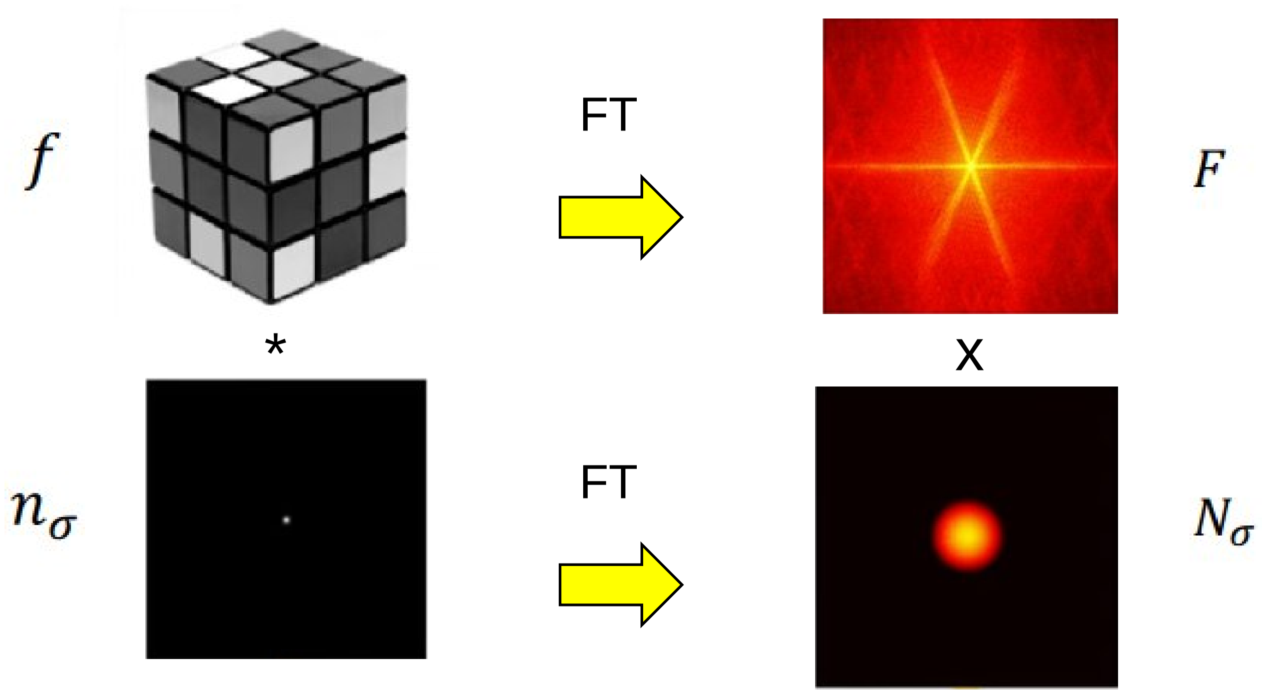
Gaussian Smoothing using Fourier Transform

f

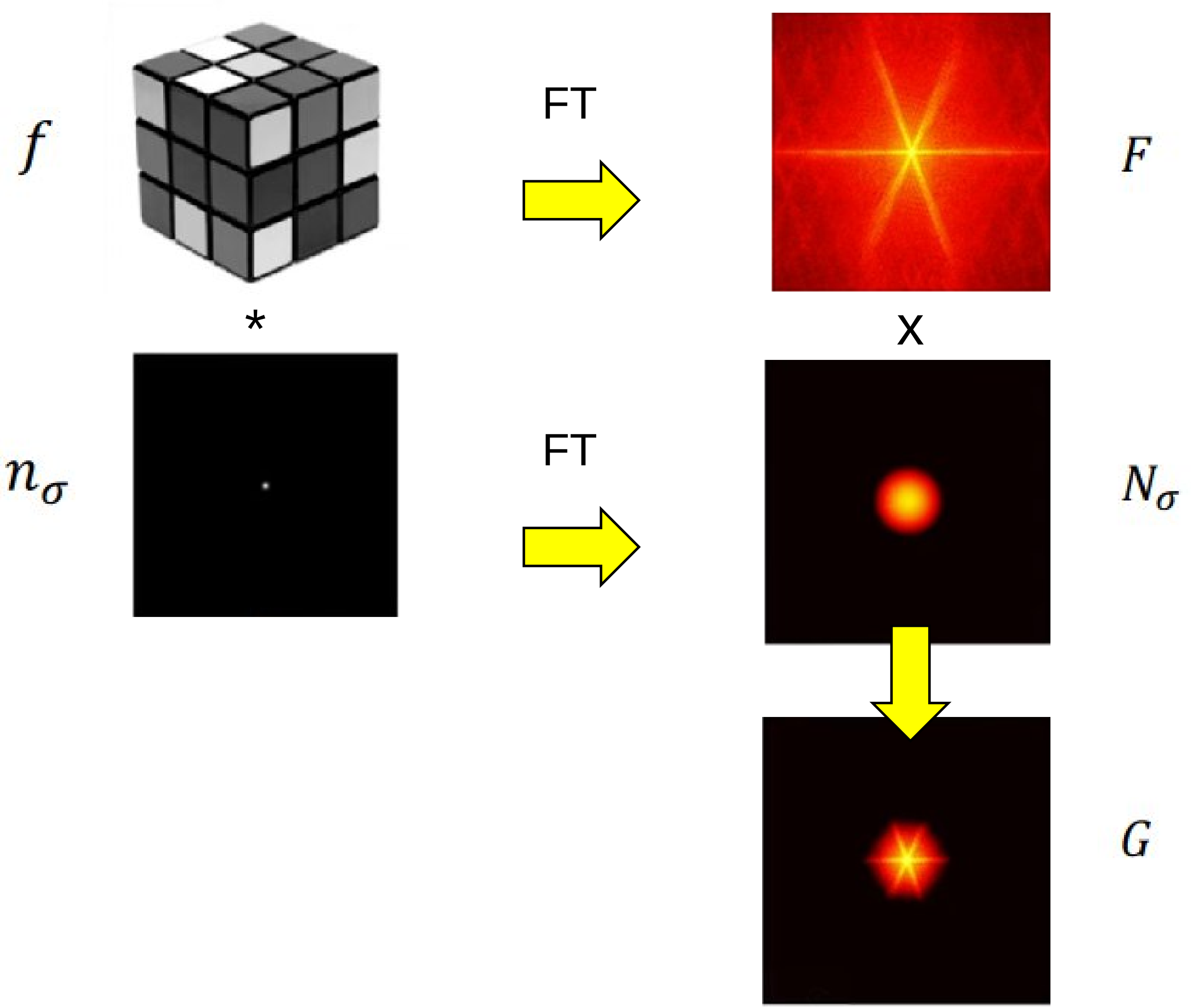
*

n_σ

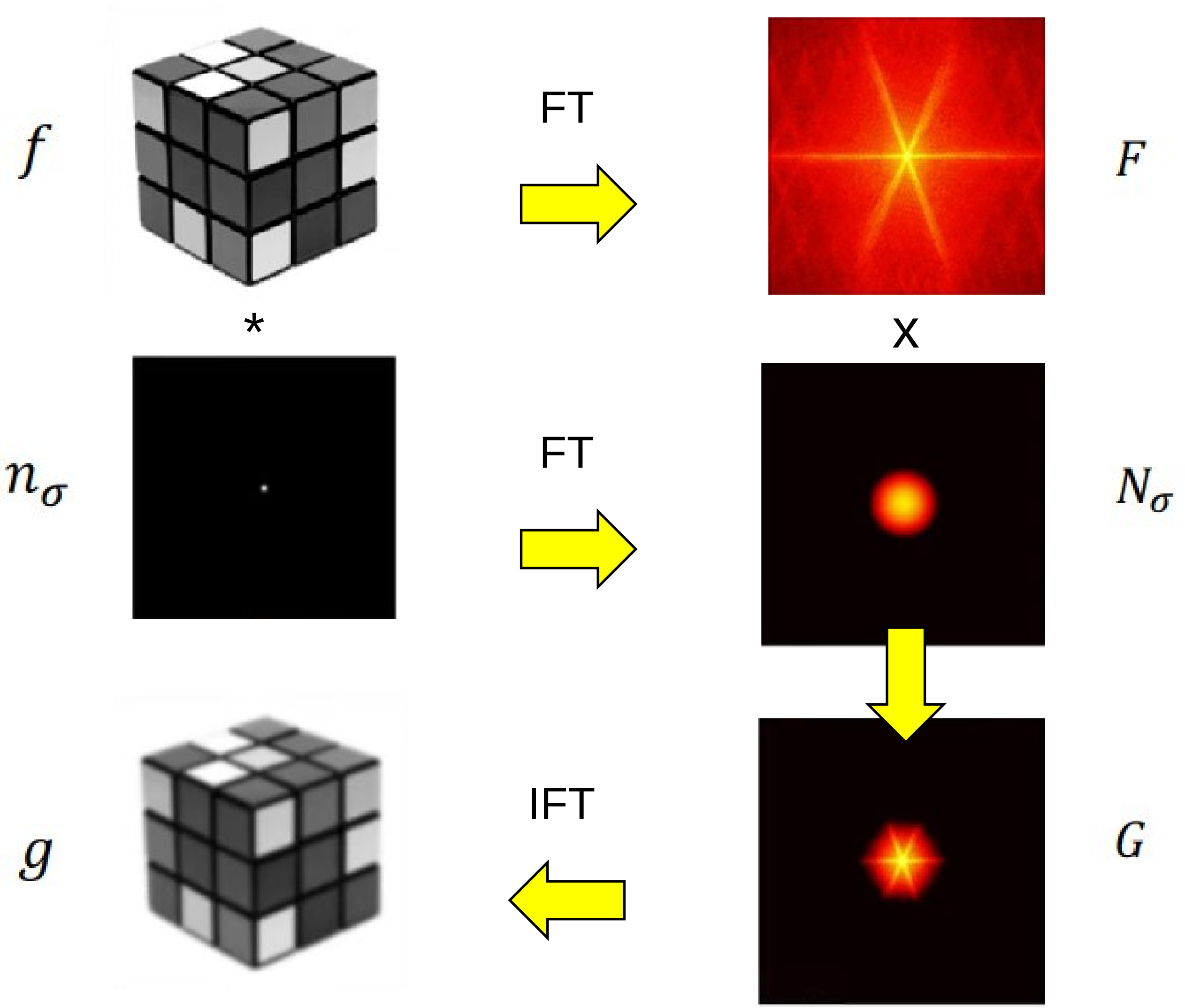
Gaussian Smoothing using Fourier Transform



Gaussian Smoothing using Fourier Transform



Gaussian Smoothing using Fourier Transform



Importance of phase

FT

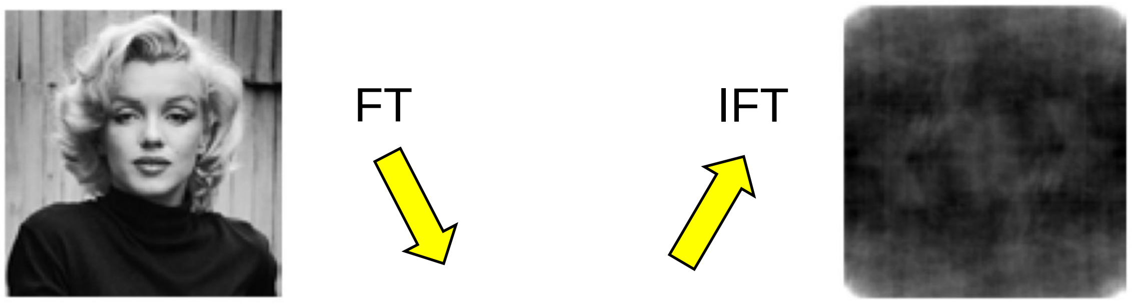
IFT

Original Image

Magnitude Preserved,
Phase Set to Zero

OPPENHEIM, Alan V. et LIM, Jae S. The importance of phase in signals. *Proceedings of the IEEE*, 1981, vol. 69, no 5, p. 529-541.

Importance of phase



Original Image

Magnitude Preserved,
Phase Set to Zero

Recovered image

OPPENHEIM, Alan V. et LIM, Jae S. The importance of phase in signals. *Proceedings of the IEEE*, 1981, vol. 69, no 5, p. 529-541.

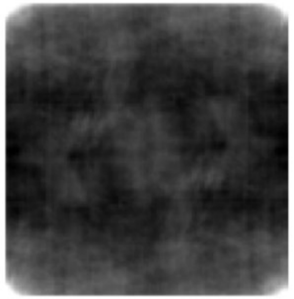
Importance of phase

Original Image

FT

Magnitude Preserved,
Phase Set to Zero

IFT



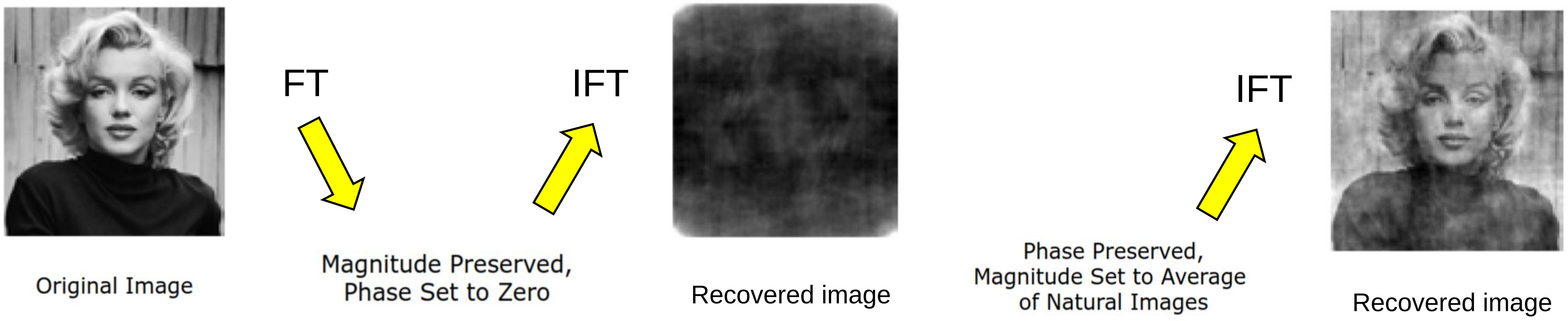
Recovered image

Phase Preserved,
Magnitude Set to Average
of Natural Images

IFT

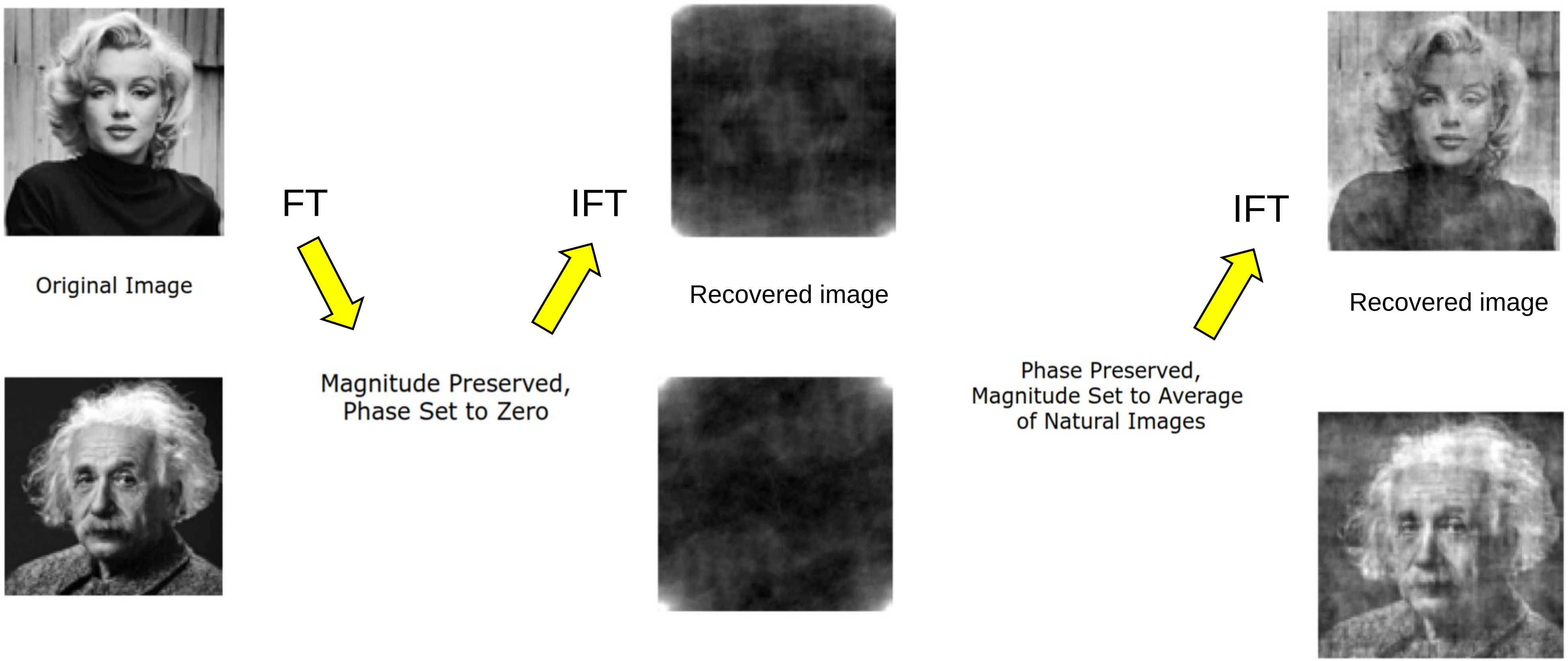
OPPENHEIM, Alan V. et LIM, Jae S. The importance of phase in signals. *Proceedings of the IEEE*, 1981, vol. 69, no 5, p. 529-541.

Importance of phase



OPPENHEIM, Alan V. et LIM, Jae S. The importance of phase in signals. *Proceedings of the IEEE*, 1981, vol. 69, no 5, p. 529-541.

Importance of phase



OPPENHEIM, Alan V. et LIM, Jae S. The importance of phase in signals. *Proceedings of the IEEE*, 1981, vol. 69, no 5, p. 529-541.

Phase vs Magnitude

Magnitude

- Represents the amplitude/strength of each frequency component
- Contains information about how much of each frequency is present; geometrical structure of features
- Shows energy distribution (bright = strong frequency, DC component brightest at center)
- Reconstructing the magnitude alone is unrecognizable, severe dynamic range problems

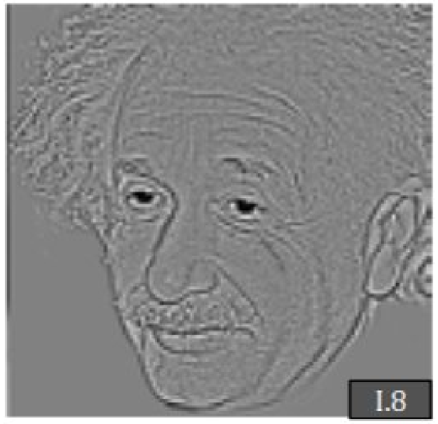
Phase

- Represents the spatial location/position of features
- Contains here edges and structures are located in the image.
- Looks like random noise when displayed, but encodes all spatial information
- Reconstructing the phase alone is degraded but somewhat recognizable

Both are required for perfect reconstruction, but phase encodes spatial locations (where features are) while magnitude encodes intensity (how strong frequencies are). Phase is typically more important for human recognition.

Hybrid images

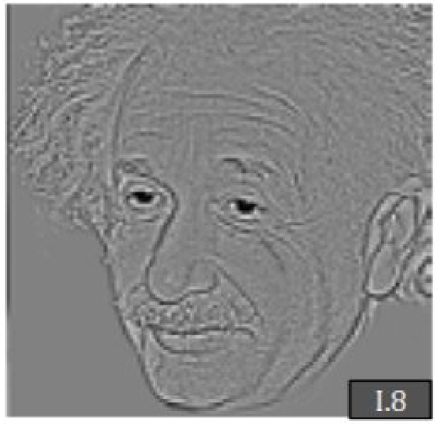
Low Freq Only



High Freq Only

Hybrid images

Low Freq Only



High Freq Only

Hybrid (Sum) Image

Hybrid images

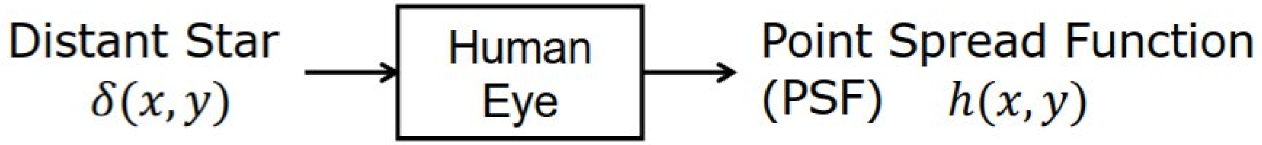
Low Freq Only

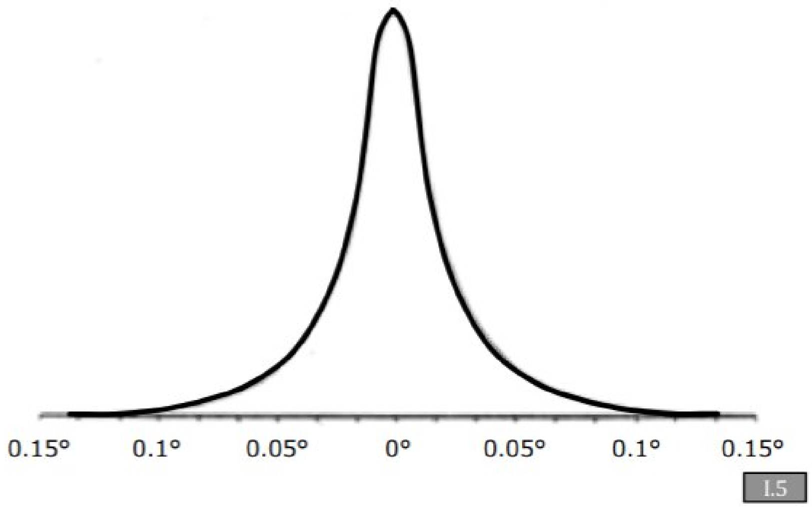
High Freq Only

Hybrid (sum) image

Why we see it that way ? PSF of Human Eye

The point-spread function (PSF) of the lens of the eye of a human eye acts as a low pass filter





Human Eye PSF